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How Complex is the Quantum Motion?

Classical Dynamics and Quantum Entanglement

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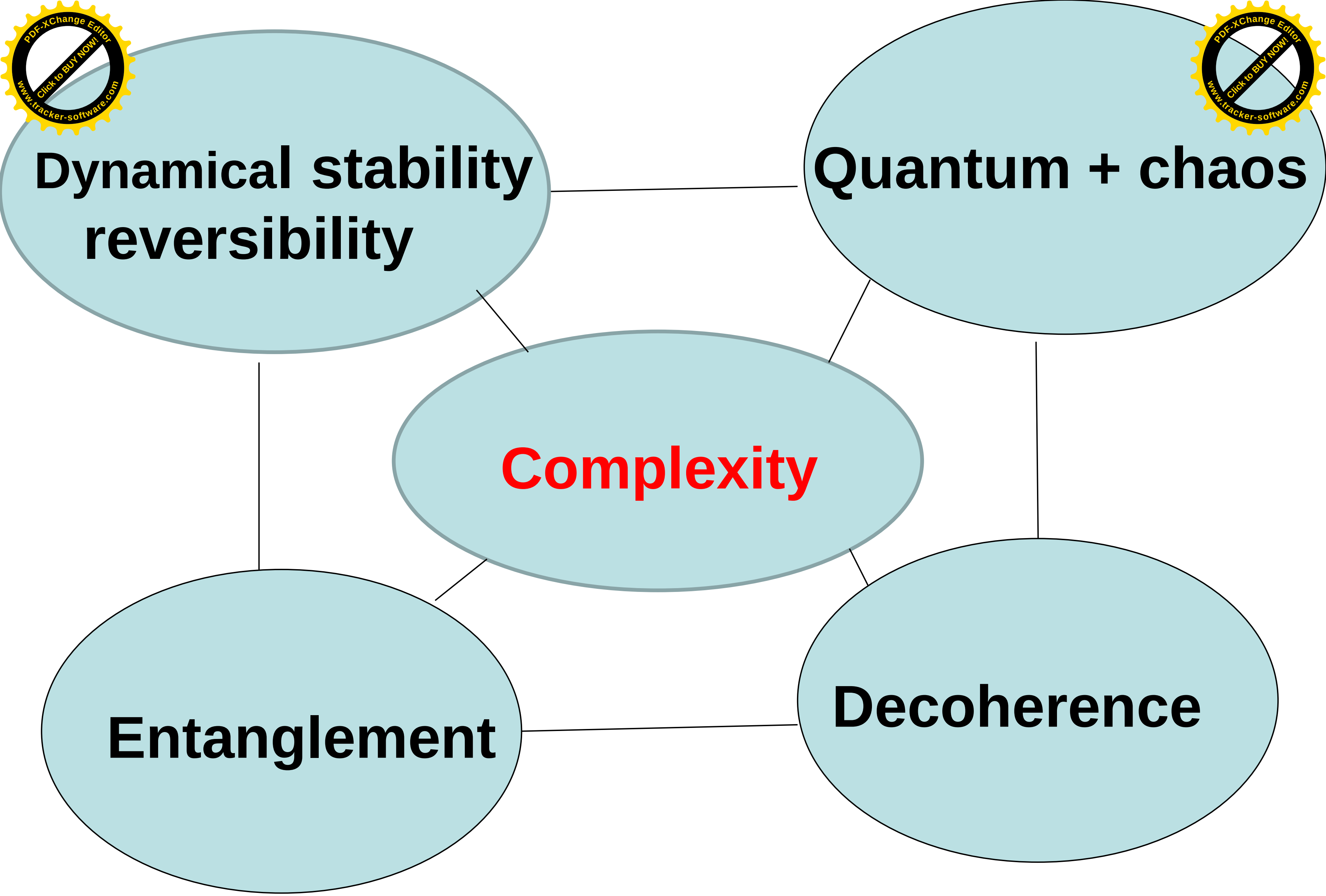
Understanding the qualitative dynamical behaviour of quantum systems is of fundamental importance in a variety of fields

Quantum dynamical complexity:

-Lack of simple description of dynamical evolution

-Loss of predictability using classical simulations


Dynamical stability
reversibility
Quantum + chaos
Complexity
Entanglement
Decoherence

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Note Title 14/06/2004

CLASSICAL CHAOS

Dynamical chaos destroys the deterministic image of classical physics ⇒ Trajectories are "random and unpredictable"

This behaviour is rooted in the <u>exponential instability</u> of motion described by lin. eqs.

$$\dot{(\delta q)} = \left(\frac{\partial^2 H}{\partial p \partial q}\right)_0 \delta q + \left(\frac{\partial^2 H}{\partial p^2}\right)_0 \delta p$$

$$\dot{(\delta p)} = \left(\frac{\partial^2 H}{\partial q^2}\right)_0 \delta q + \left(\frac{\partial^2 H}{\partial p \partial q}\right)_0 \delta p$$

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases}$$

$\delta q, \delta p \equiv$ n-dim. vectors in tangent space

The coeff. of the linear eqs. are taken on the reference trajectory ⇒ they are time-dependent.

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Local exponential instability

$$\lambda = \lim_{|t| \to \infty} \frac{1}{|t|} \ln d(t) > 0$$

$$d(t) = \sqrt{\delta q^2 + \delta p^2}$$

$\equiv$ length of tangent vector

In order to predict a new segment of a chaotic trajectory one needs an additional information proportional to the length of this segment and independent of the previous length

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Randomness parameter r

G.C., B. Chirikov
"Quantum chaos"
Cambridge Univ. press
1995

$$r = \frac{\lambda\,|t|}{|\ln \mu|}$$

$\lambda \equiv$ KS entropy

$\mu \equiv$ accuracy at which a trajectory is recorded

- Prediction is possible inside the finite interval $r < 1$.

- For $r > 1$ the motion is not distinguishable from a completely random motion

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Classical description can be given in terms of distribution functions (instead of trajectories).

Distribution functions obey the linear Liouville eq.

Mixing $\Rightarrow$ statistical relaxation to steady state.

Relaxation is time reversible (as for trajectories).

However it is non recurrent (while the motion on trajectories, integrable or chaotic, recurs infinitely many times).

How to characterize the complexity of a quantum state?

The notion of complexity in classical mechanics cannot be directly transferred to quantum Mechanics (no notion of trajectories)

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The phase space approach can be used for both classical and quantum mechanics.

Compare phase space distributions

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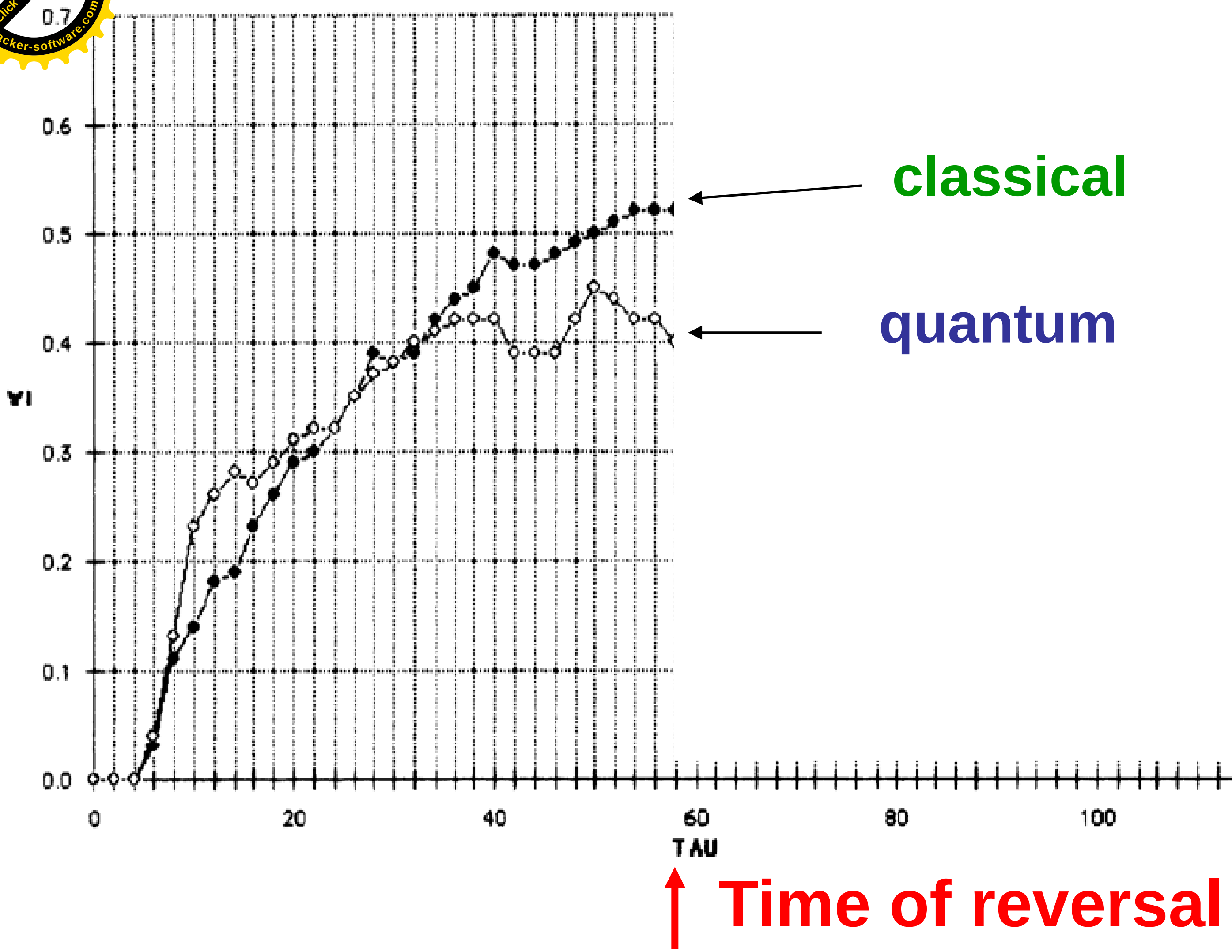
HYDROGEN ATOM IN EXTERNAL MICROWAVE FIELD



$$H = -1/2n^2 + \varepsilon z(n,\theta)\cos\omega t$$

prl 56, 2437 (1986)

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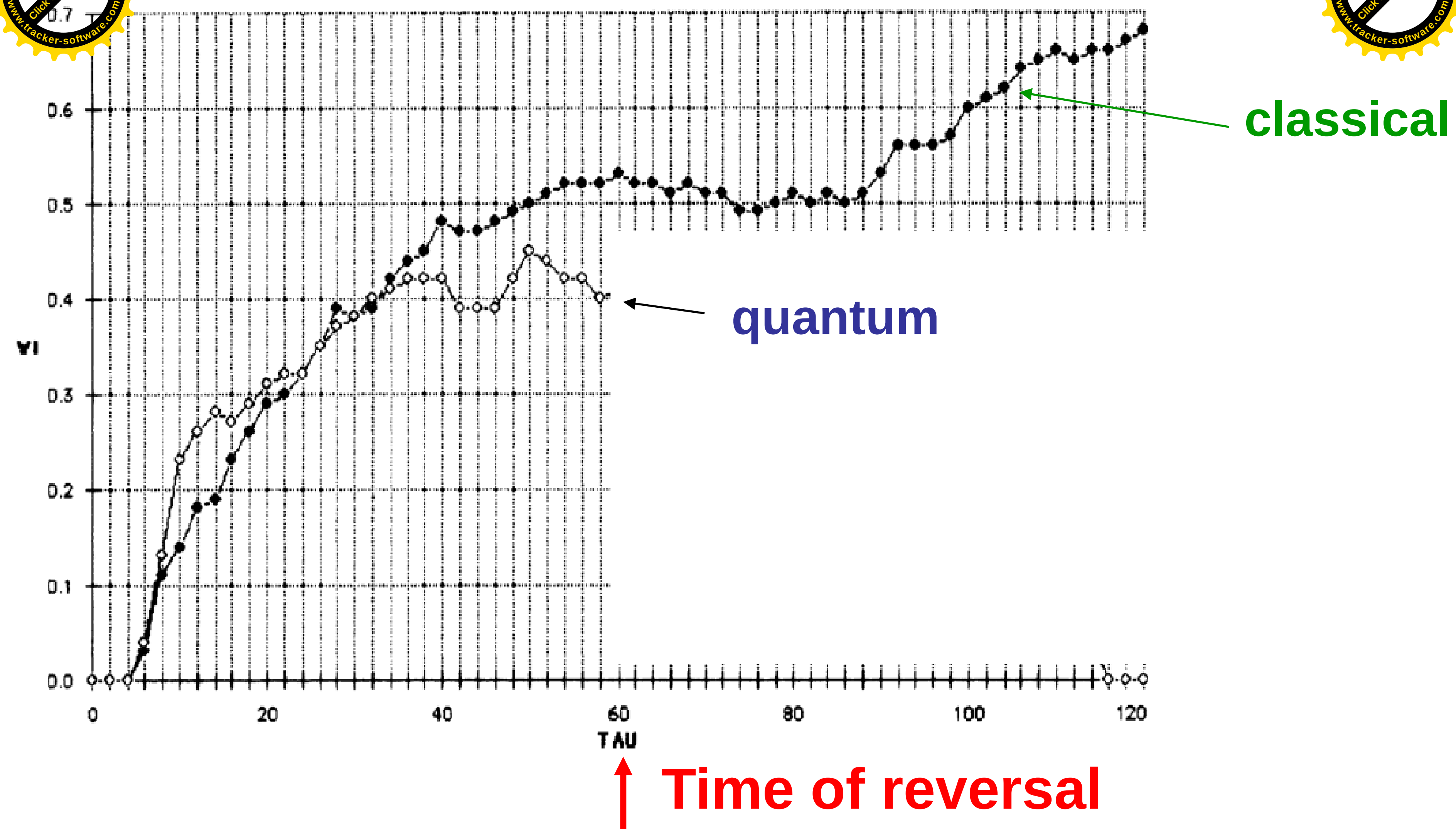

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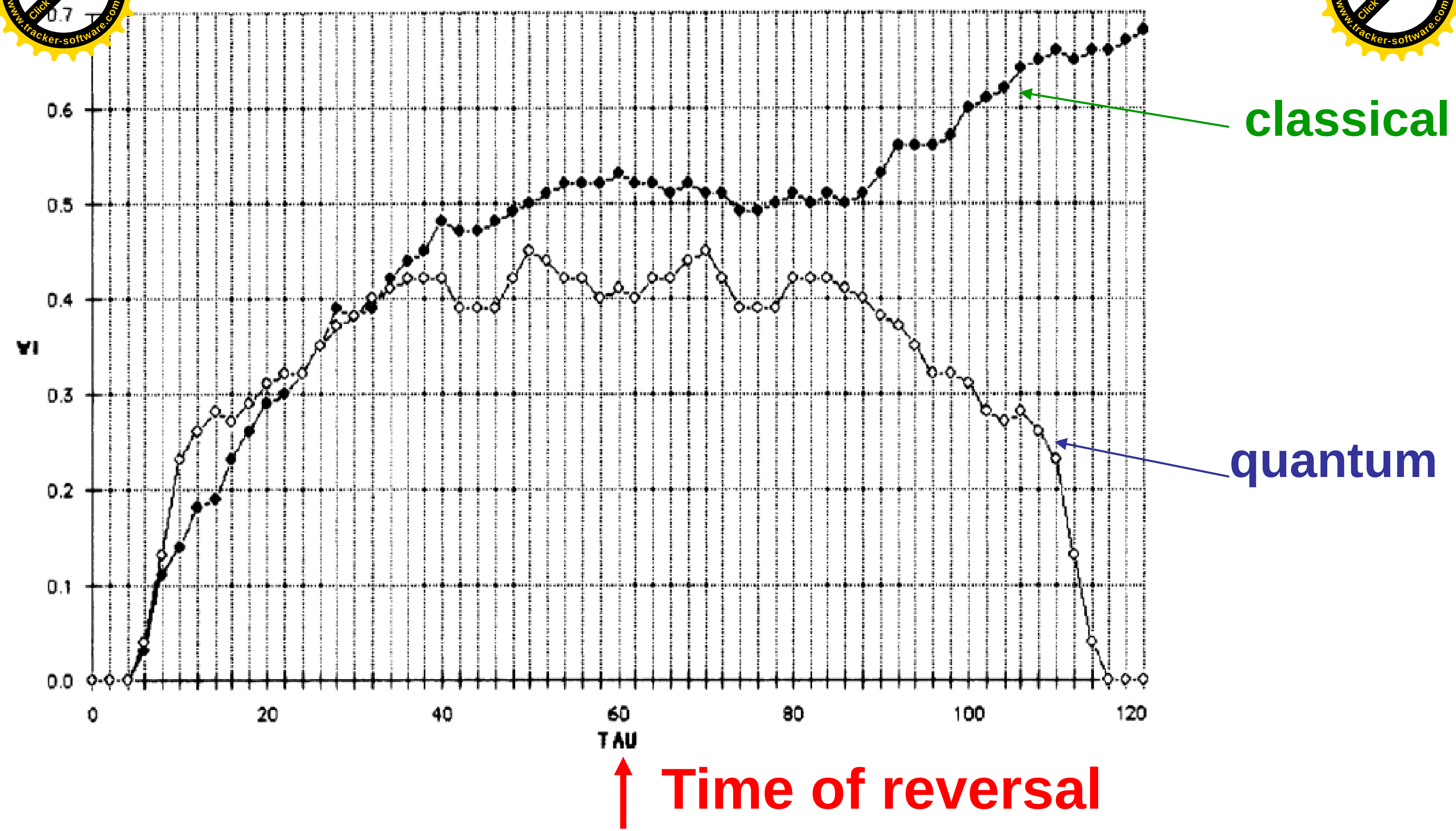

$$H = -1/2n^2 + \varepsilon z(r,\theta)\cos\omega t$$

prl 56, 2437 (1986)

In classical mechanics the number of Fourier components of the classical distribution function in phase space grows, in time:

- **linearly for integrable systems**

- **exponentially for chaotic systems,**

The growth rate of the number of harmonics-which is determined by Lyapunov exponent- is a measure of classical complexity.(smaller and smaller scales are explored exponentially fast with time).

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Let us consider the number of angular harmonics that is the number of terms with appreciable large amplitudes $W^{(cl)}{}_m(I;t)$ in the expansion of the classical distribution function $W^{(cl)}(\alpha^*, \alpha; t)$ over the eigenfunctions of the angular momentum operator $e^{im\theta}$

$$\langle m^2 \rangle_t \propto e^{t/\tau_c}$$

The number grows exponentially

The number of harmonics of the Wigner function is the natural quantity to measure the complexity of a quantum state

the number of the components of the Wigner function at any given time is related to the degree of excitation of the system.

Unrestricted growth of this number is not physical

Such exponential growth cannot take place in quantum mechanics: the Fourier components of the Wigner function are related to expectation values of physical observables (Chirikov et al 1981)

$$\langle n \rangle_t \simeq \langle I \rangle_t / \hbar$$

Exponential growth is possible only up to $\bar{t}$

$$e^{\lambda \bar{t}} \simeq \langle I \rangle_t / \hbar$$

That is

$$\bar{t} = t_E \simeq \frac{1}{\lambda} \ln \frac{\langle I \rangle_t}{\hbar}$$

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Quantum dynamics in phase space

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Allows to compare evolution of Wigner function with classical evolution of phase space density

Write the hamiltonian in terms of a set of bosonic creation-annihilation operators:

$$\hat{H}(\hat{a}_1^\dagger, \ldots, \hat{a}_N^\dagger, \hat{a}_1, \ldots, \hat{a}_N; t)$$

$$\equiv \hat{H}^{(0)}(\hat{n}_1, \ldots, \hat{n}_N) + \hat{H}^{(1)}(\hat{a}_1^\dagger, \ldots, \hat{a}_N^\dagger, \hat{a}_1, \ldots, \hat{a}_N; t),$$

$$[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0, \; [\hat{a}_i^\dagger, \hat{a}_j] = \delta_{ij}$$

$$\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$$ number operators

$|\alpha\rangle = |\alpha_1\alpha_2\cdots\alpha_N\rangle$ $\hat{a}_i|\alpha_i\rangle = (\alpha_i/\sqrt{\hbar})|\alpha_i\rangle$

$$W(\boldsymbol{\alpha},\boldsymbol{\alpha}^*;t) = \frac{1}{\pi^N}\sum_{\mathbf{m}} W_{\mathbf{m}}(\mathbf{I};t)e^{i\mathbf{m}\cdot\boldsymbol{\theta}}$$

where $\mathbf{m}, \mathbf{I}, \boldsymbol{\theta}$ are N-dimensional vectors, whose components $I_k \geq 0$, $0 \leq \theta_k < 2\pi$ are defined by the relations $\alpha_k = \sqrt{I_k}e^{-i\theta_k}$, with $k = 1, \ldots, N$. Here, I_k and θ_k can be regarded as our quantum phase-space variables, analogous to the action and angle variables in the classical phase space. Note that

$$\langle \mathbf{m}^2 \rangle_t = \sum_{\mathbf{m}} \mathbf{m}^2 \mathcal{W}_{\mathbf{m}}(t)$$

$$\mathcal{W}_{\mathbf{m}}(t) \equiv \frac{\int d\mathbf{I}|W_{\mathbf{m}}(\mathbf{I};t)|^2}{\sum_{\mathbf{m}}\int d\mathbf{I}|W_{\mathbf{m}}(\mathbf{I};t)|^2}$$

$$S(t) = -\sum_{m_1,\ldots,m_N \geq 0} \mathcal{W}_{\mathbf{m}}(t) \ln[\mathcal{W}_{\mathbf{m}}(t)]$$

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The main computational advantage of the above c-number $\boldsymbol{\alpha}$-phase-space approach is that the Wigner function's harmonics $\mathcal{W}_{\mathbf{m}}$ can be computed very conveniently from the density matrix written in the basis of the eigenvectors $|\mathbf{n}\rangle = |n_1 \cdots n_N\rangle$ of the unperturbed Hamiltonian $\hat{H}^{(0)}$. Indeed,

$$\mathcal{W}_{\mathbf{m}}(t) = \frac{\sum_{\mathbf{n}} |\langle \mathbf{n} + \mathbf{m} | \hat{\rho}(t) | \mathbf{n} \rangle|^2}{\sum_{m_1,\ldots,m_N \geq 0} \sum_{\mathbf{n}} |\langle \mathbf{n} + \mathbf{m} | \hat{\rho}(t) | \mathbf{n} \rangle|^2}$$

Example: a kicked quartic oscillator model

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$$\hat{H} = \hbar\omega_0 \hat{n} + \hbar^2 \hat{n}^2 - \sqrt{\hbar}\, g(t)(\hat{a} + \hat{a}^\dagger)$$

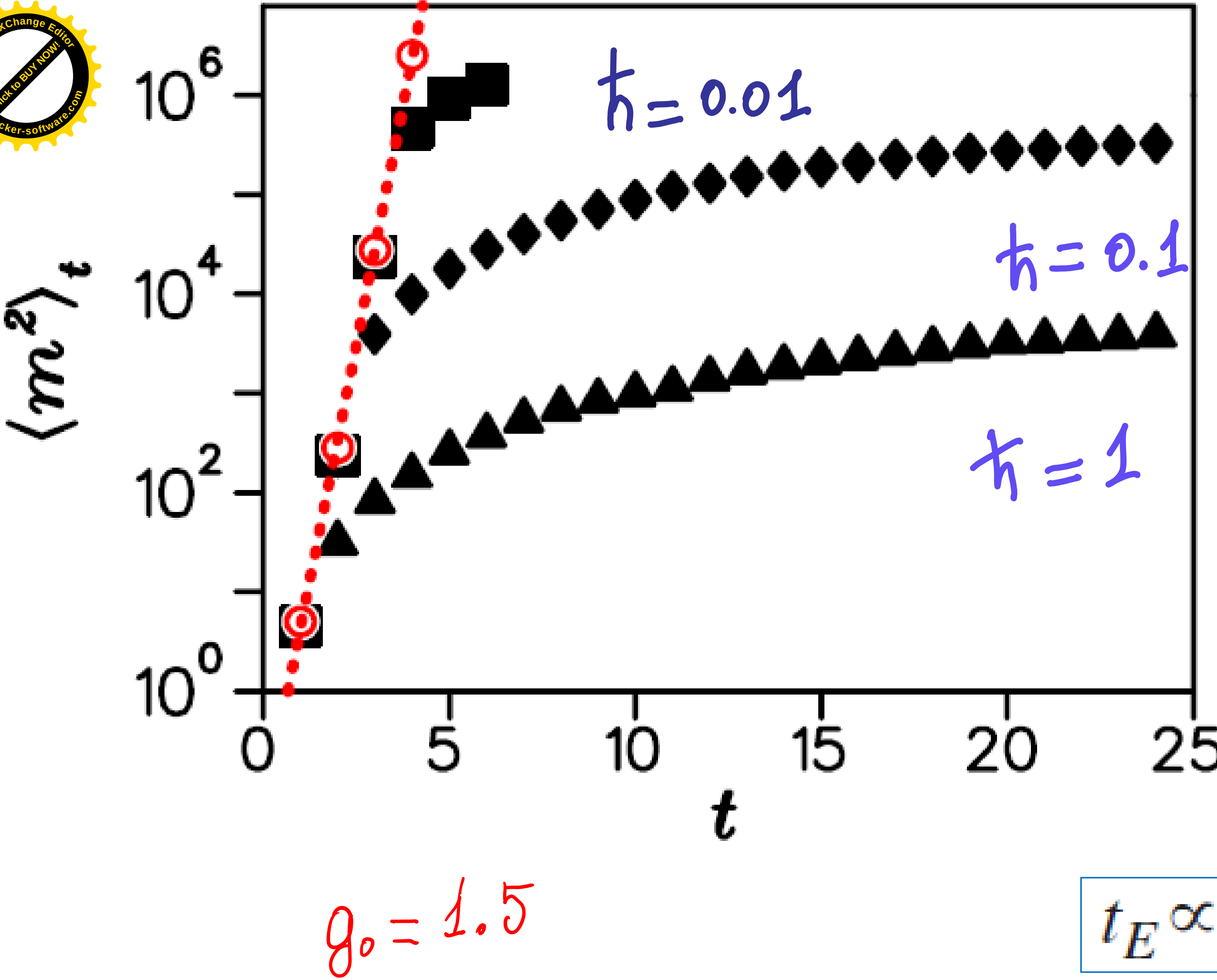
$$g(t) = g_0 \Sigma_s \delta(t - s)$$

$$\hat{n} = \hat{a}^\dagger \hat{a},\ [\hat{a}, \hat{a}^\dagger] = 1$$

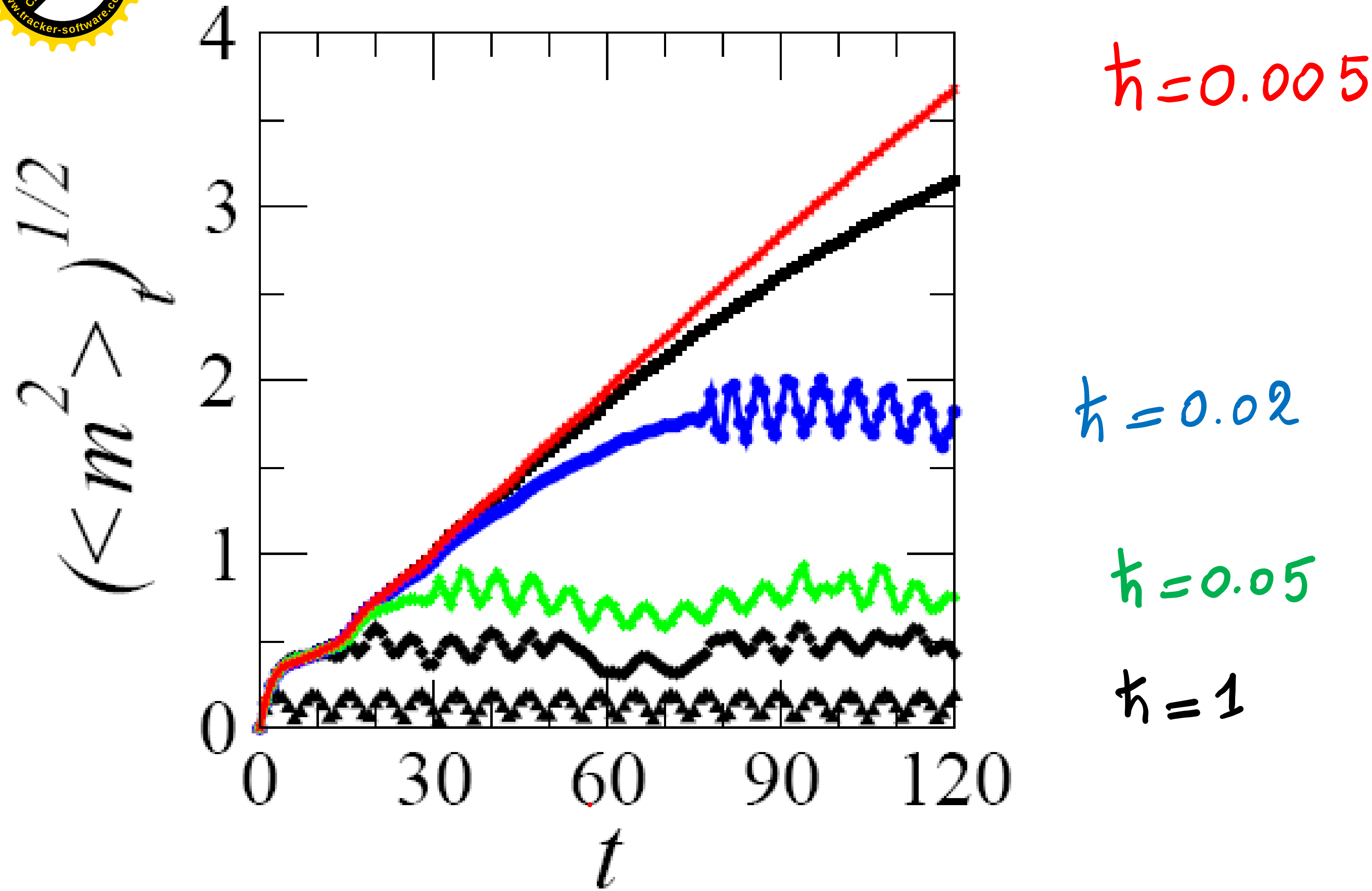
$$H_c = \omega_0 |\alpha|^2 + |\alpha|^4 - g(t)(\alpha^* + \alpha)$$

Classical Hamiltonian

Chaotic for $g_0 \gtrsim 1$

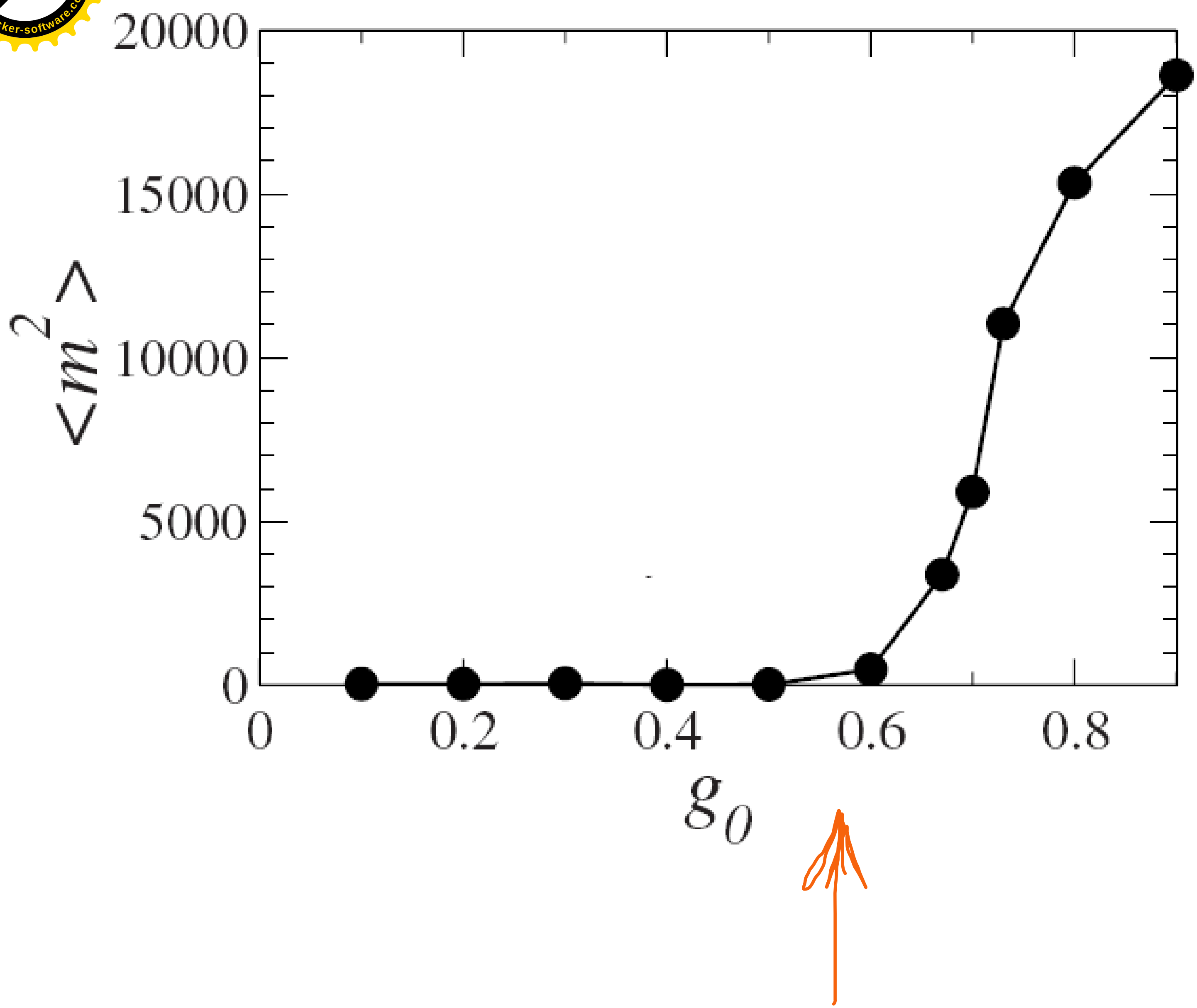


$$t_E \propto \ln \hbar$$



Integrable case

$$t_H \propto \hbar^{-1}$$



$\hbar = 0.01$

$t = 3$

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Many-body quantum systems

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Ising chain of N spins in a tilted magnetic field

$$\hat{H} = J\sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z + \sum_i [h_x \hat{\sigma}_i^x + h_z \hat{\sigma}_i^z] \qquad \hbar = J = 1$$

where J is the spin-spin coupling constant.

$\hat{\sigma}_i^\alpha$ Pauli operators

h_x and h_z fields amplitudes along x and z

$h_x = 0$, $h_z = 0$ integrable limits

$h_z = 0$ corresponds to Ising model in transverse magnetic field and has quantum phase transition at $J = h_x$

Using Schwinger boson representation the spin Hamiltonian is mapped onto an interacting boson Hamiltonian

$$\hat{H} = J\sum_{i=1} (\hat{a}_i^\dagger \hat{a}_i - \hat{b}_i^\dagger \hat{b}_i)(\hat{a}_{i+1}^\dagger \hat{a}_{i+1} - \hat{b}_{i+1}^\dagger \hat{b}_{i+1})$$

$$+ \sum_{i=1} [h_x(\hat{a}_i^\dagger \hat{b}_i + \hat{b}_i^\dagger \hat{a}_i) + h_z(\hat{a}_i^\dagger \hat{a}_i - \hat{b}_i^\dagger \hat{b}_i)]$$

$$\hat{H}^{(0)} = J\sum_i^N \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z + \sum_i^N h_z \hat{\sigma}_i^z, \qquad \hat{H}^{(1)} = \sum_i h_x \hat{\sigma}_i^x$$

integrable

Transition to quantum chaos

For $h_x = h_z = J$ fully chaotic

$$|\Psi_{\text{in}}\rangle = |\downarrow\downarrow\cdots\downarrow\rangle$$

initial, pure state

only one component is excited $\Rightarrow$ $S(0) = 0$

$$S(t) = -\sum_{m_1,\ldots,m_N \geq 0} W_{\mathbf{m}}(t)\ln[W_{\mathbf{m}}(t)]$$

For a system of N spins only 2^N values are possible

Maximum value S= N ln(2)

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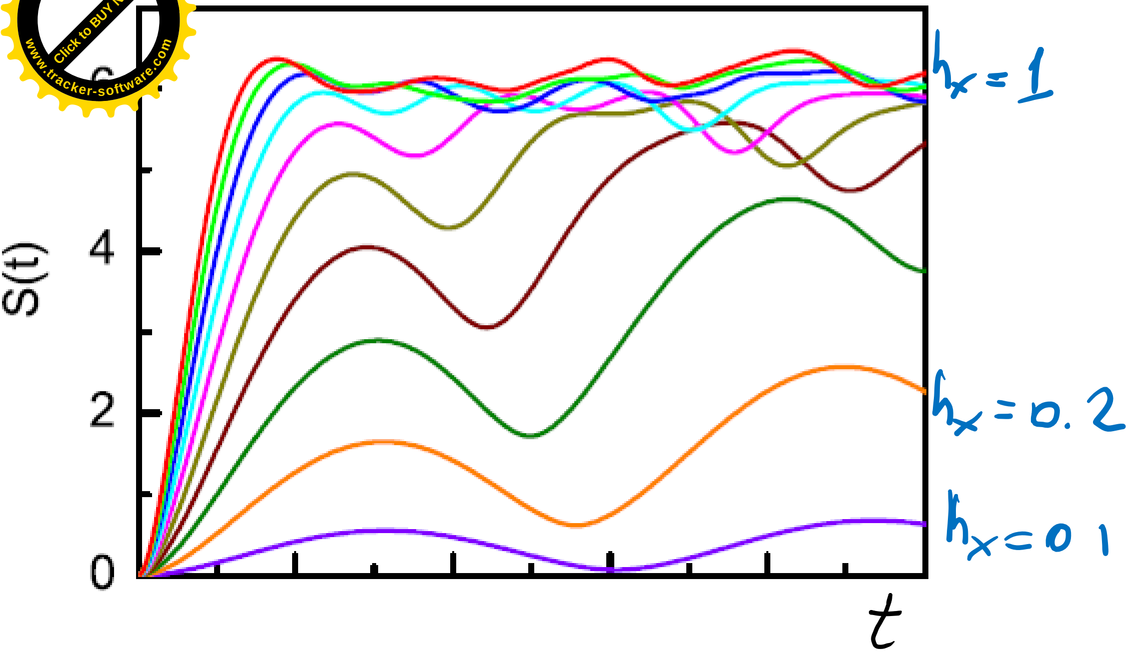

NON INTEGRABLE

Chain of N=10 spins

$h_z = 1.0$

Maximum value
S = N ln(2) ≈ 6.9

The linear increase of S(t) implies an exponential growth of the number of harmonics

Consider the Ising chain in transverse magnetic field:

$$\hat{H}^{(0)} = J\sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z, \quad \hat{H}^{(1)} = \sum_i h_x \hat{\sigma}_i^x$$

Integrable

We might expect a linear increase of the number of harmonics corresponding to logarithmic growth of S(t).

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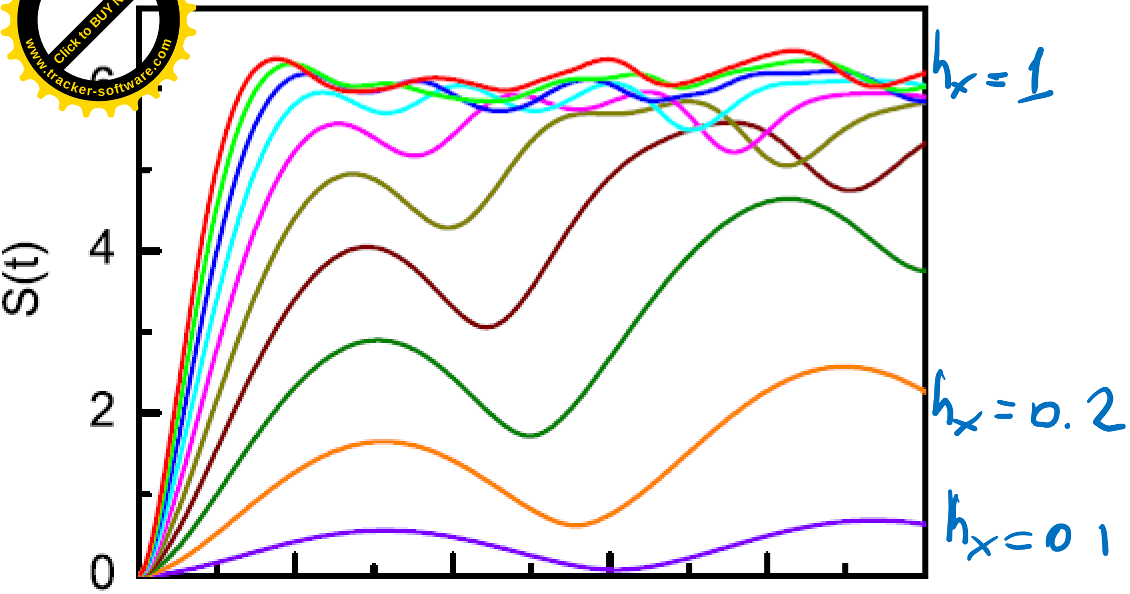
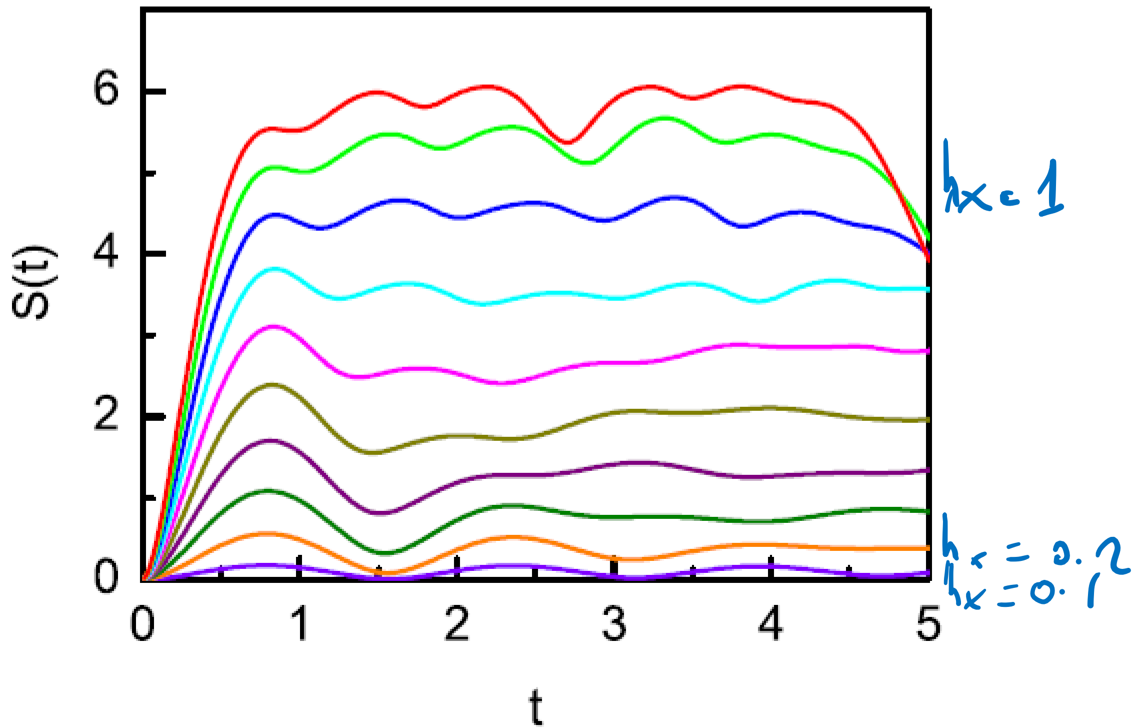


NON INTEGRABLE



INTEGRABLE

Lack of distinction between integrable and non-integrable is it due to entanglement generation?

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A state in $\mathcal{H}$ is entangled or non separable if it cannot be written as a simple tensor product of a state $|\alpha\rangle_1$ belonging to $\mathcal{H}_1$ and a state $|\beta\rangle_2$ belonging to $\mathcal{H}_2$

The state $|\psi\rangle = |\alpha\rangle_1 \otimes |\beta\rangle_2$ is separable

$$\frac{1}{\sqrt{2}}\left(|01\rangle + |11\rangle\right)$$

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

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$$\frac{1}{\sqrt{2}}\left(|01\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes |1\rangle$$

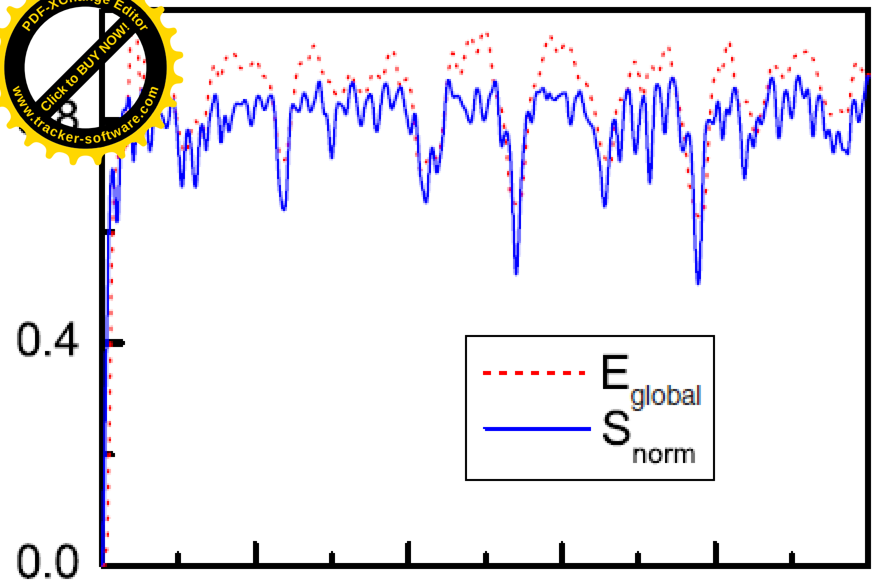
$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \neq \left(a|0\rangle + b|1\rangle\right) \otimes \left(c|0\rangle + d|1\rangle\right)$$

compare S(t) with the so-called Global entanglement

$$E_{\text{global}} = 2\left(1 - \frac{1}{N}\sum_{k=1}^{N} \mathrm{Tr}[\hat{\rho}_k^2]\right)$$

where $\hat{\rho}_k$ is the density matrix of the kth spin after tracing over all other spins in the system. E_{global} is the average bipartite entanglement over all possible bipartitions between a single qubit and the rest of the system. It is easy to see that $0 \le E_{\text{global}} \le 1$. Values of E_{global} close to 1 indicate highly entangled many-body states. When a many-body state is not entangled, E_{global} equals to zero.

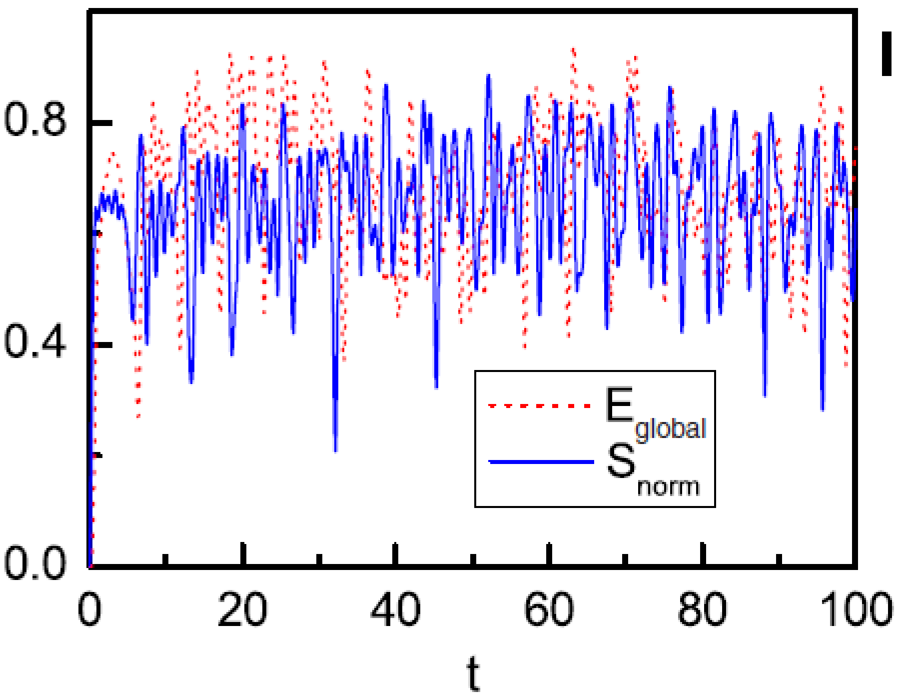
Chain of 10 spins $h_x = 0.8$



Non Int.

$h_z = 1.0$

$S_{norm} = \frac{S}{N \ln 2}$



Int.

The entropy measure (related to phase space complexity) reflects the degree of multipartite entanglement

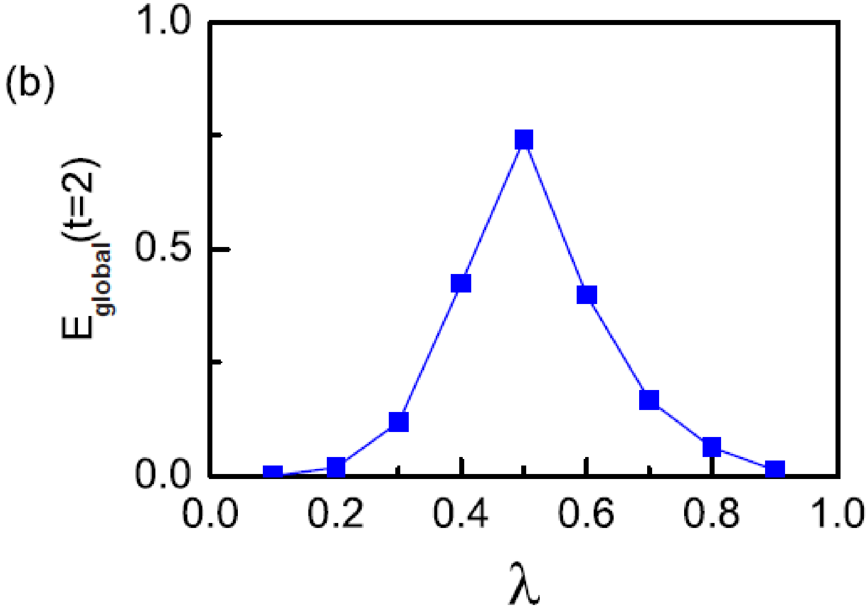
S(t) near a quantum critical point

Transverse Ising chain

$$\hat{H}^{(0)} = J\sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z, \quad \hat{H}^{(1)} = \sum_i h_x \hat{\sigma}_i^x$$

$h_x = 1$ **critical point**

$$\lambda \equiv h_x / (J + h_x)$$



S(t) near a quantum critical point

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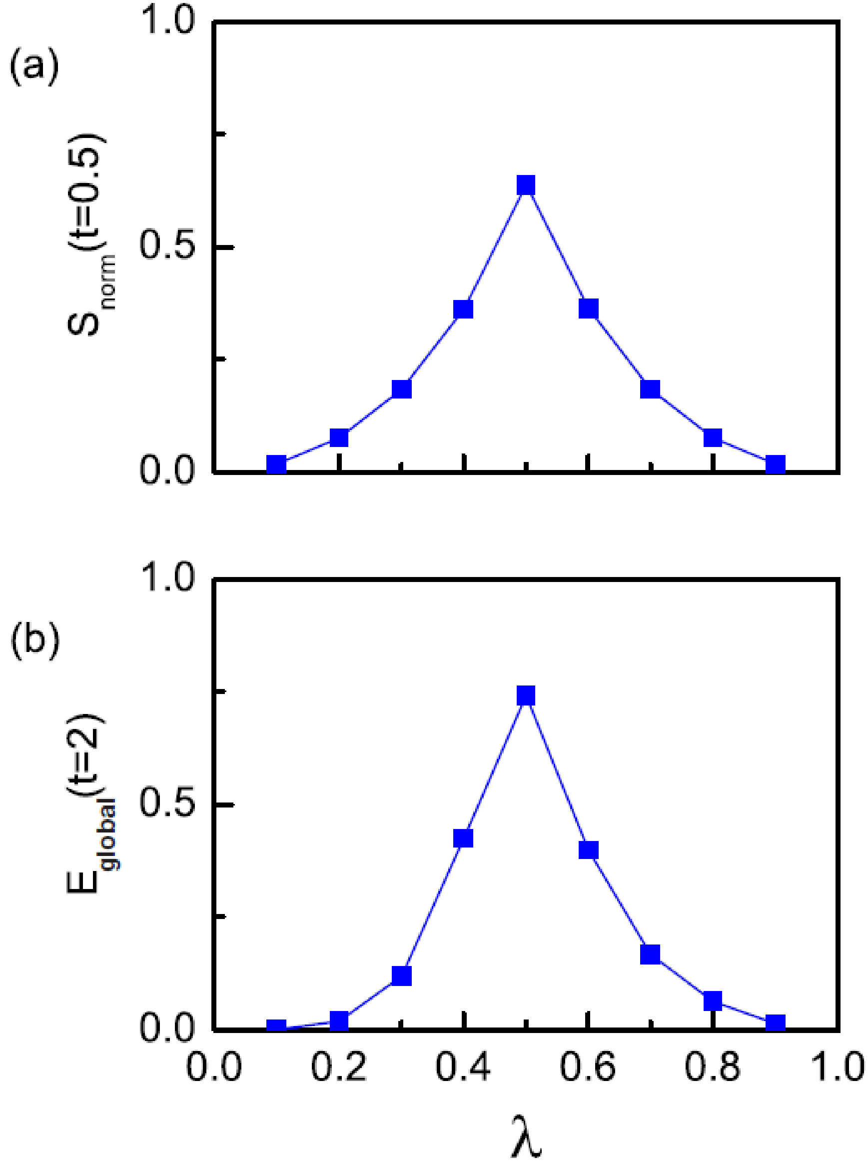

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$h_x = 1$ **critical point**

$$\lambda \equiv h_x / (J + h_x)$$

A measure of growth rate of number of harmonics exhibits a sharp peak at

$\lambda = 1/2$

We propose an entropy measure *S(t) for* many-body quantum dynamical complexity

-The measure is illustrated in the example of the Ising chain in a homogeneous tilted magnetic field.

-in both integrable and chaotic regimes the number of harmonics of the Wigner function grows exponentially with time.

The observed exponential growth of Wigner harmonics in the many-body quantum integrable regime must be attributed to a source of complexity absent in classical dynamics, that is, entanglement

Classical Dynamics and Quantum Entanglement

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Entanglement and classical dynamics

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S. Chaudhury[1], A. Smith[1], B. E. Anderson[1], S. Ghose[2] & P. S. Jessen[1]

Nature 461, 768 (2009)

"…present experimental evidence for dynamical entanglement as a signature of chaos"

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"…present experimental evidence for dynamical entanglement as a signature of chaos"

nature physics | LETTERS
PUBLISHED ONLINE: 11 JULY 2016 | DOI: 10.1038/NPHYS3830

Ergodic dynamics and thermalization in an isolated quantum system

C. Neill[1*†], P. Roushan[2†], M. Fang[1†], Y. Chen[2†], M. Kolodrubetz[3], Z. Chen[1], A. Megrant[2], R. Barends[2], B. Campbell[1], B. Chiaro[1], A. Dunsworth[1], E. Jeffrey[2], J. Kelly[2], J. Mutus[2], P. J. J. O'Malley[1], C. Quintana[1], D. Sank[2], A. Vainsencher[1], J. Wenner[1], T. C. White[2], A. Polkovnikov[3] and J. M. Martinis[1,2]

Kiched top

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Spin angolar momentum of a single Cs atom in the F=3 hyperfine ground state.

a spin F governed by:

$$H = \hbar p F_y \sum_{n=0}^{\infty} f(t - n\tau) + \hbar \frac{\kappa}{2F\tau} F_x^2$$

We use as a starting point for our kicked-top experiments an ensemble of laser-cooled Cs atoms prepared by optical pumping in a desired spin-coherent state $\rho_0 \approx |\theta,\phi\rangle\langle\theta,\phi|$. In a given run of the experiment, each member of the ensemble is subjected to n periods of the kicked-top Hamiltonian, and the entire density operator for the final state is experimentally reconstructed[19]. The process is repeated

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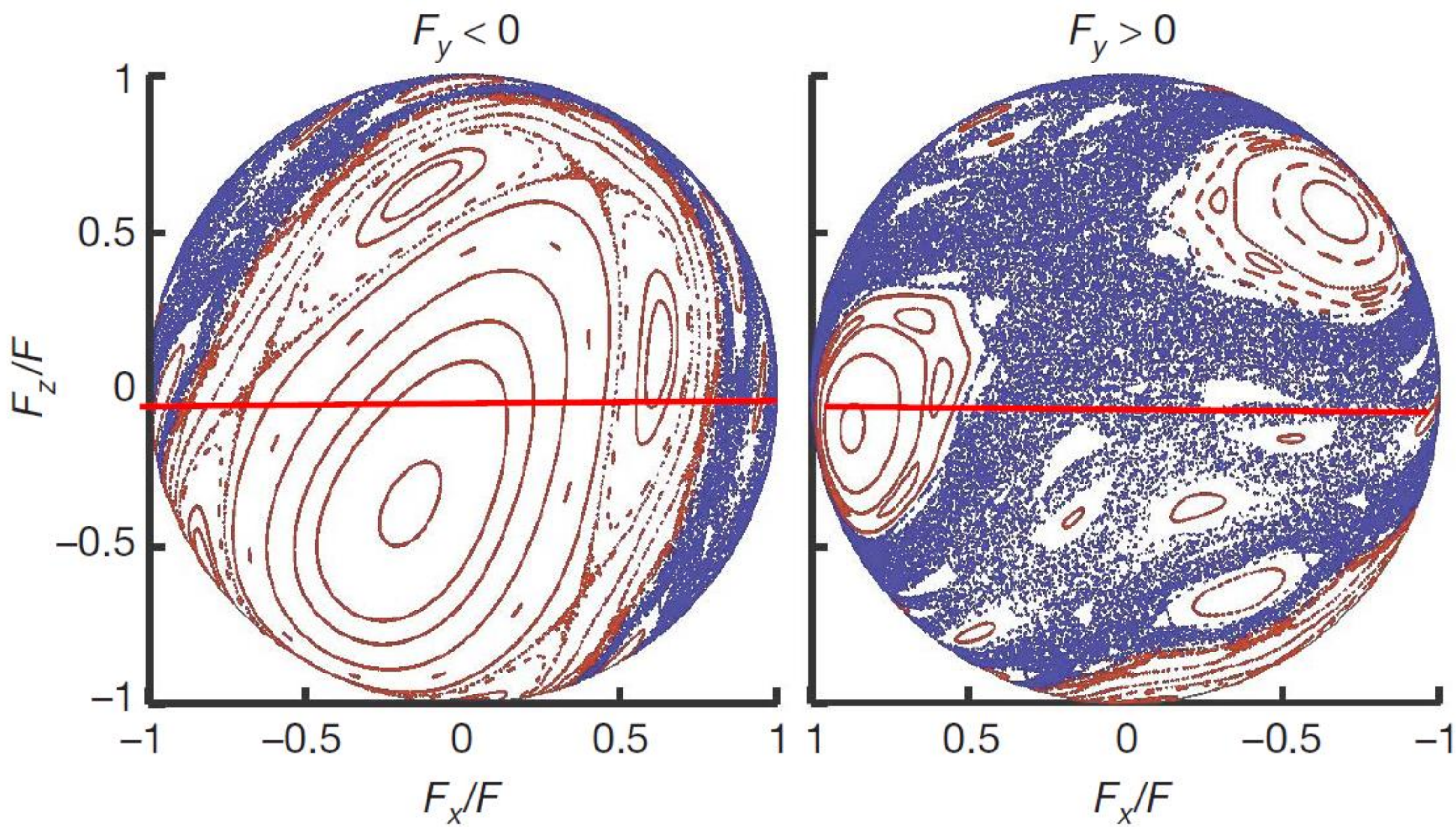
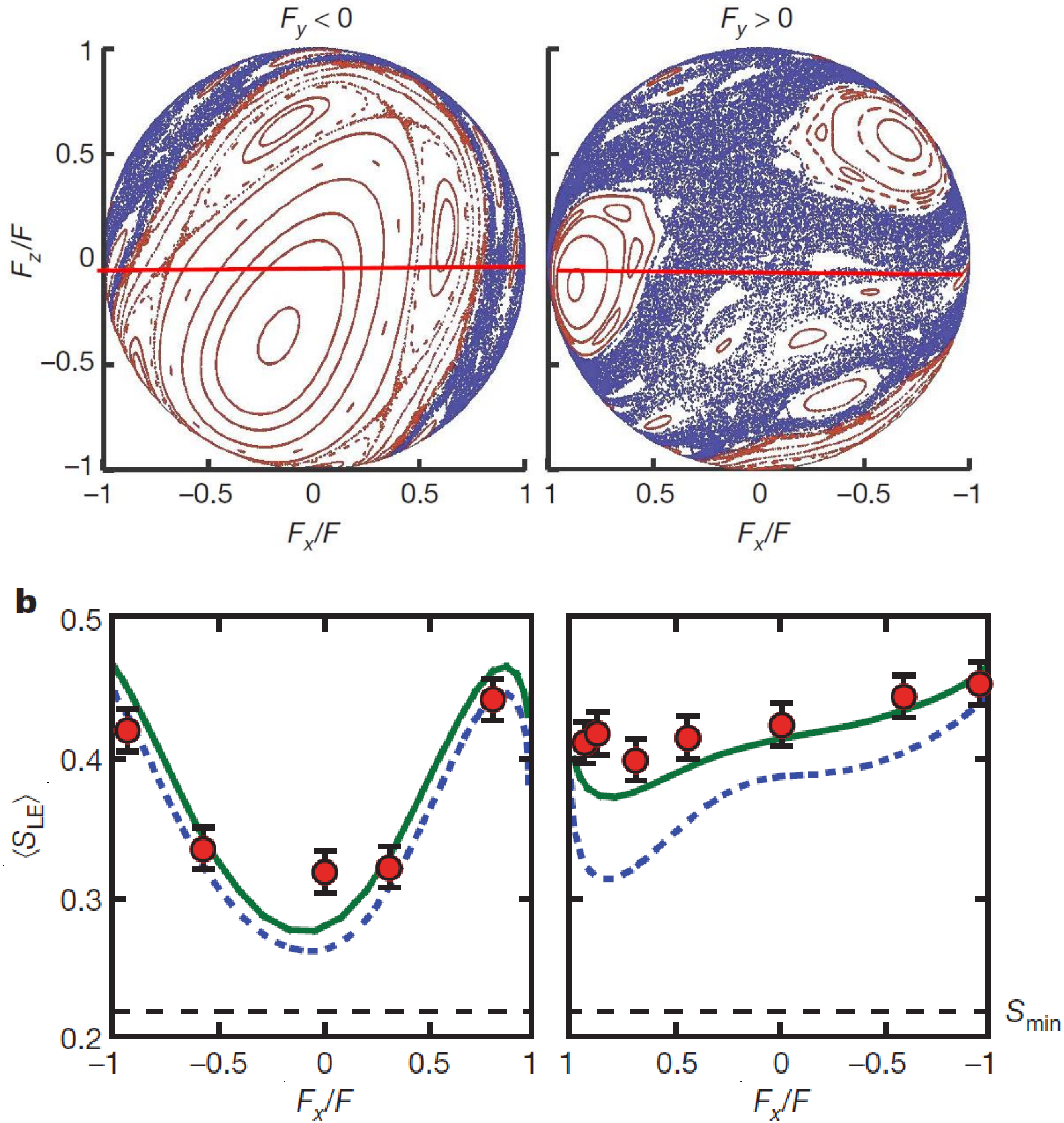
The atomic spin is the sum of electron and nuclear spins F=S+I

ρ_e = Reduced density operator for the electron spin

$$S_{\mathrm{LE}} = 1 - \mathrm{Tr}\left[\rho_e^2\right]$$





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ρ_e: Reduced density operator for the electron spin

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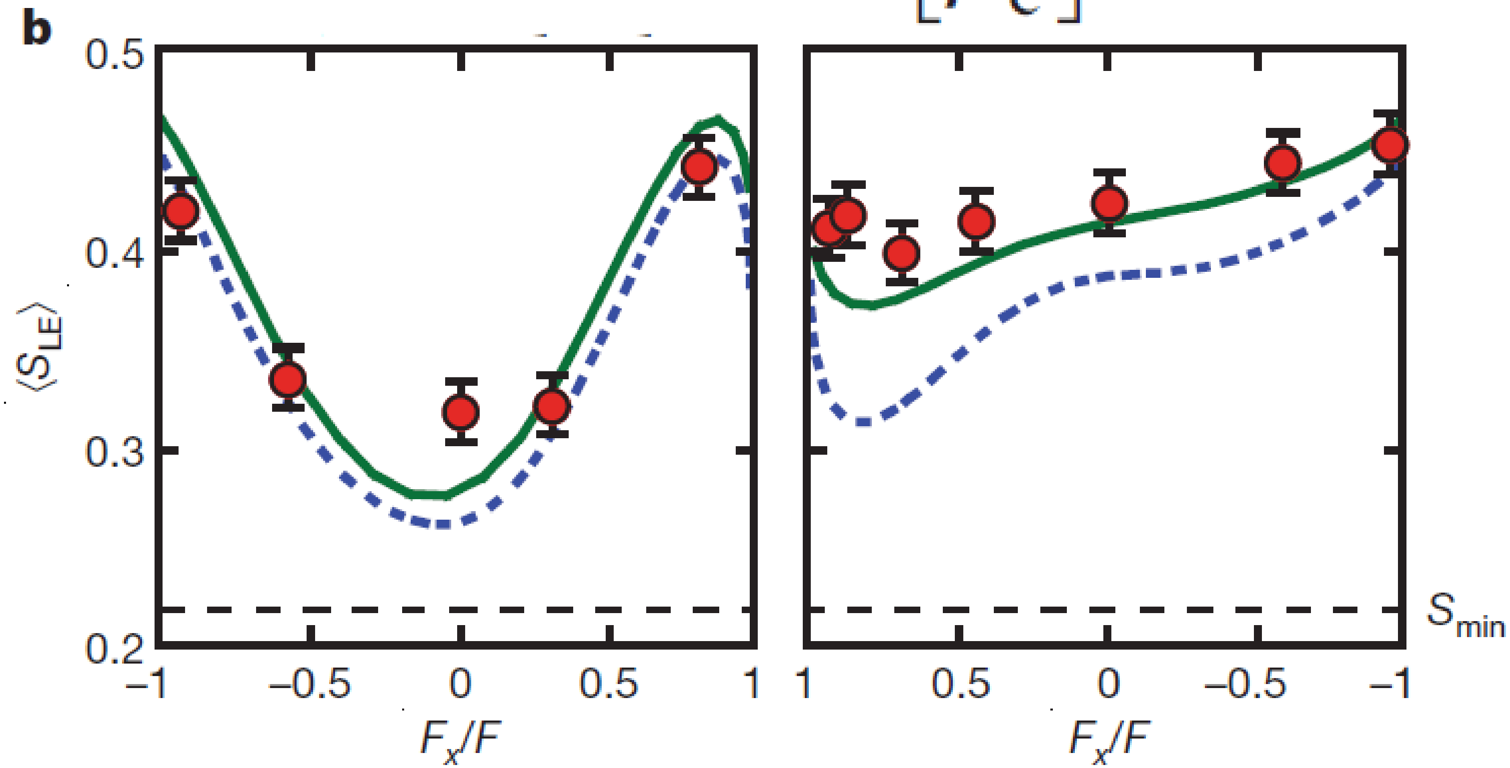
$$S_{LE} = 1 - \mathrm{Tr}\left[\rho_e^2\right]$$



"….This is the first experimental evidence that the purely quantum property of entanglement is a good signature of classical chaos." Nature 461, 768 (2009)

The Toda lattice

$$\hat{H} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + e^{-\hat{q}_1} + e^{-(\hat{q}_2 - \hat{q}_1)} + e^{\hat{q}_2} - 3.$$

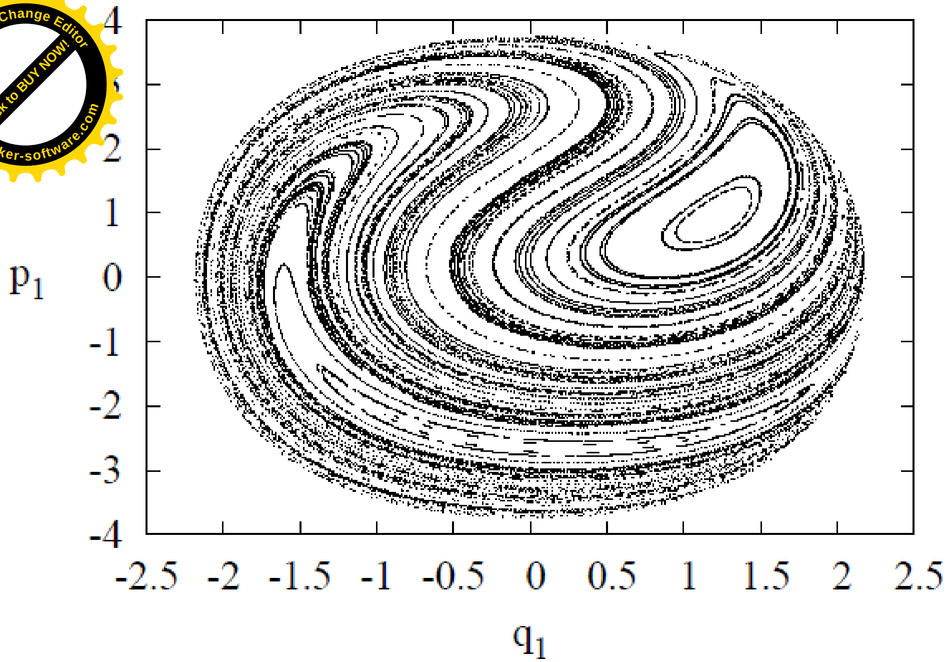
PHYSICAL REVIEW A VOLUME 12, NUMBER 4 OCTOBER 1975

Stochastic transition in the unequal-mass Toda lattice*

Giulio Casati
Istituto di Fisica, Via Celoria 16, 20133 Milano, Italy

Joseph Ford
School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332

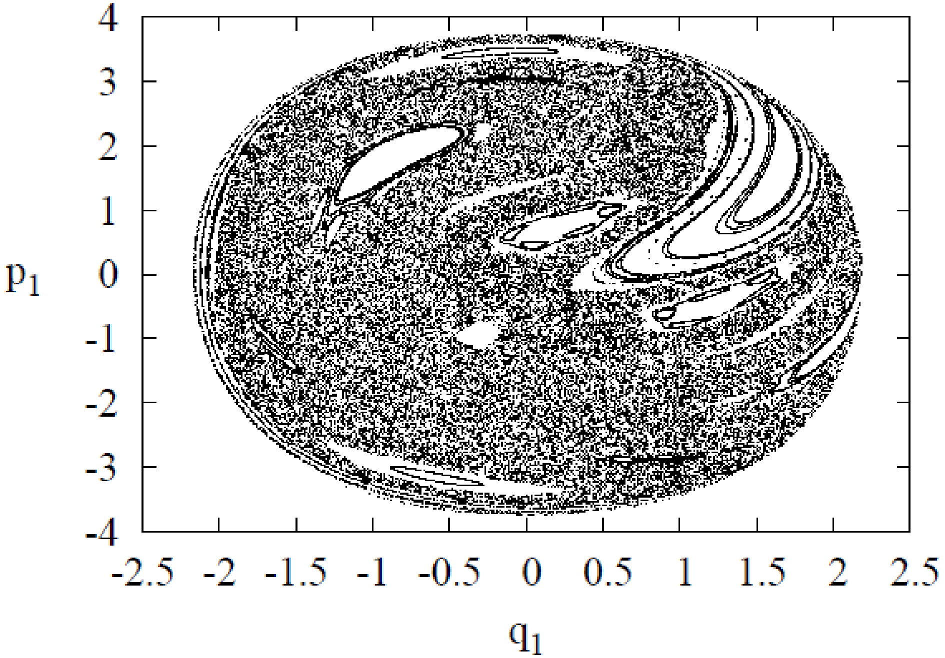
(Received 5 June 1975)



Integrable

$m_1 = m_2 = 1$

E=7



Mixed phase space

$m_1 = 1, m_2 = 0.54$

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Starting from an inital coherent ,separable, state of the two particles $|Q_1, P_1, Q_2, P_2\rangle$ we compute $|\psi(t)\rangle$

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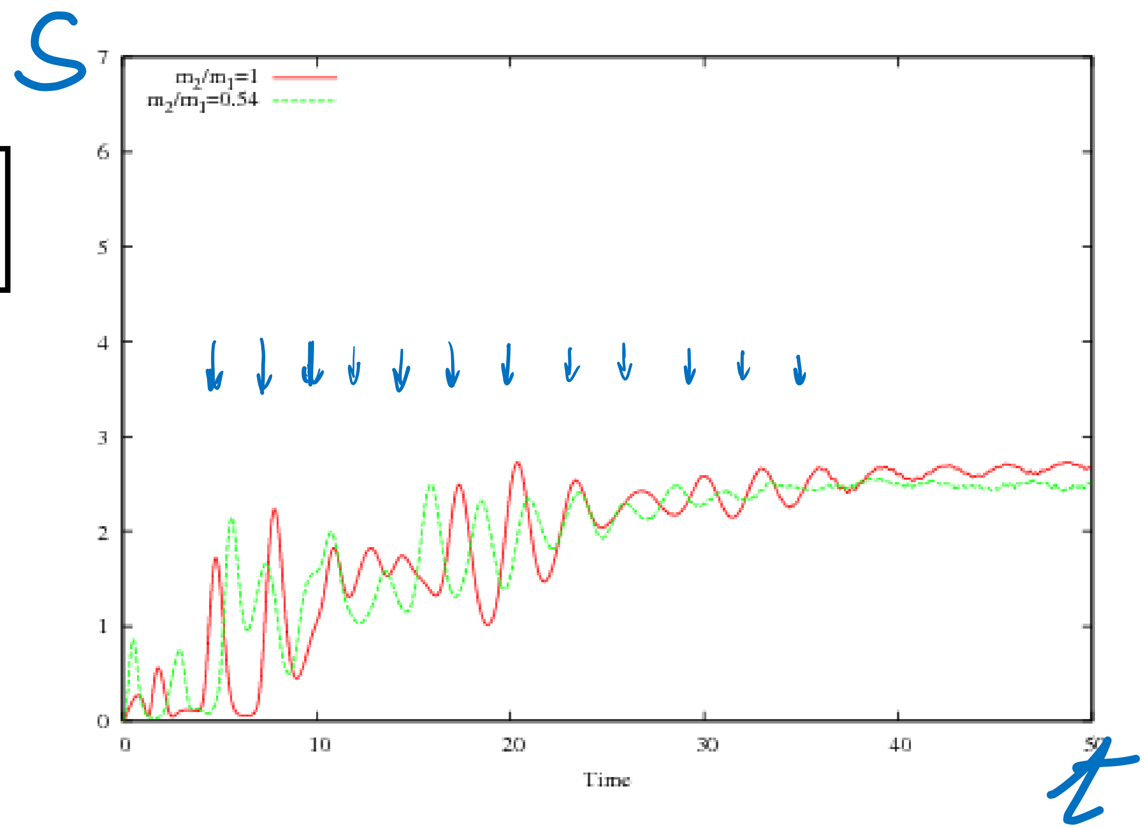
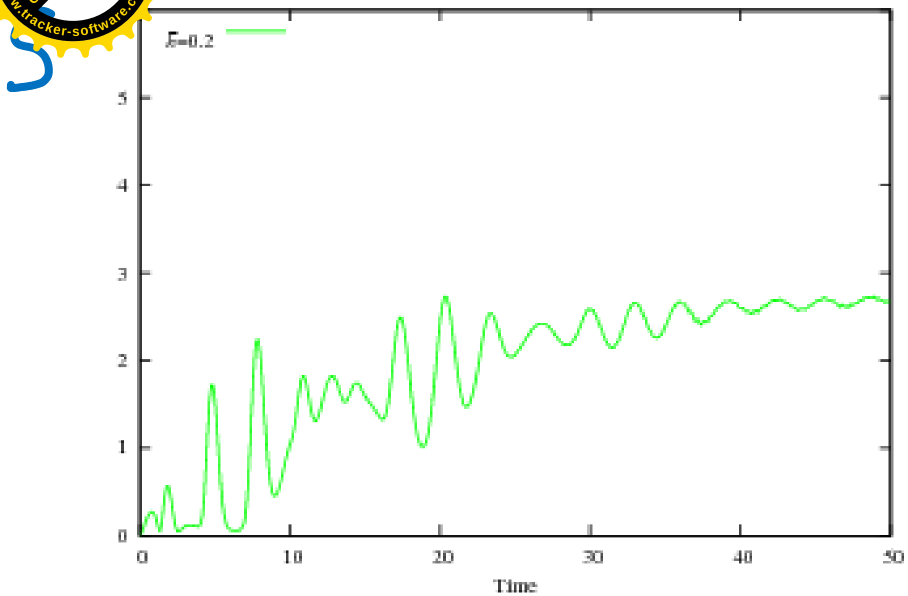
$$\hat{\rho}_1(t) = \sum_{n_2} \langle n_2|\psi(t)\rangle\langle\psi(t)|n_2\rangle$$

$$S = -Tr(\hat{\rho}_1 \log \hat{\rho}_1)$$

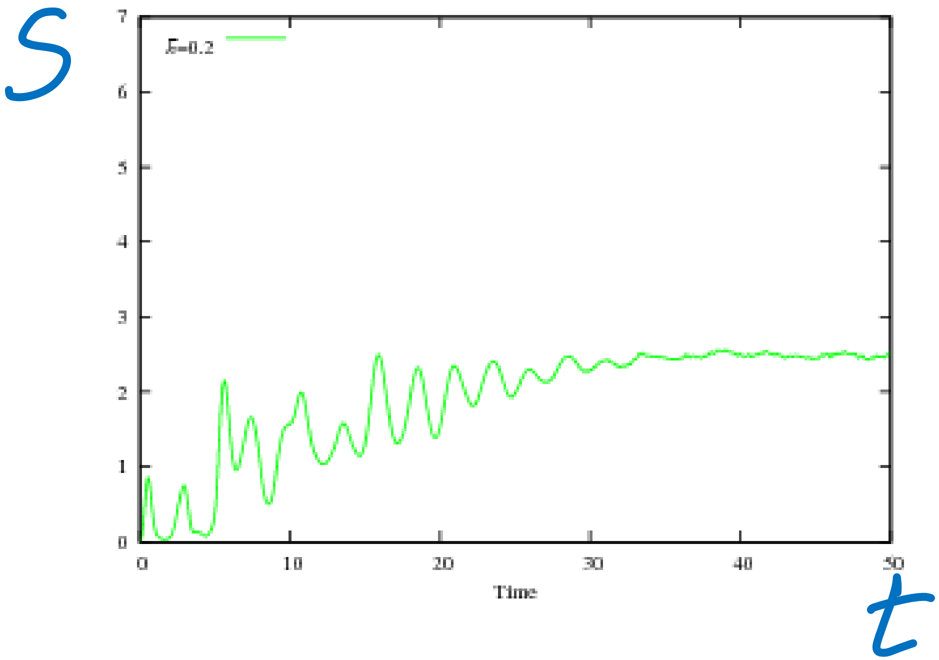


E=7 $\hbar$ **= 0.2**



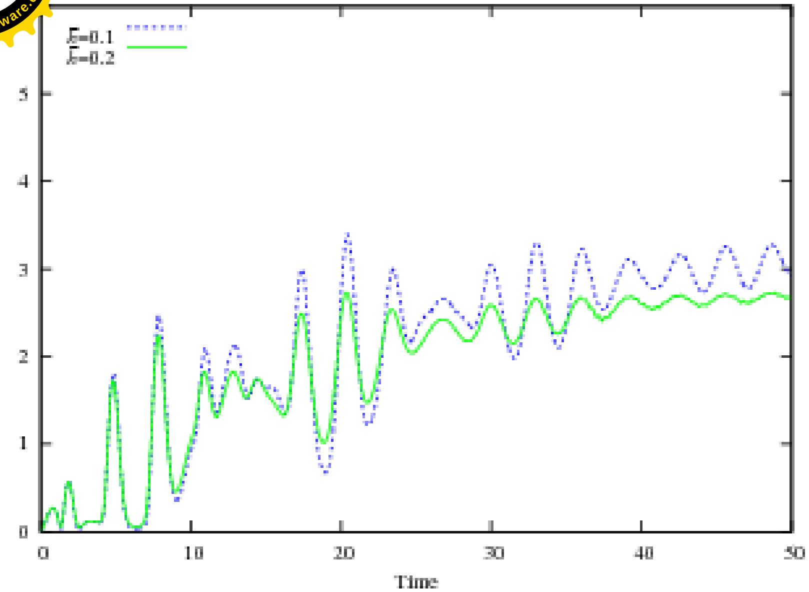
Integrable

E=7 **ħ= 0.2**



Chaotic

Integrable

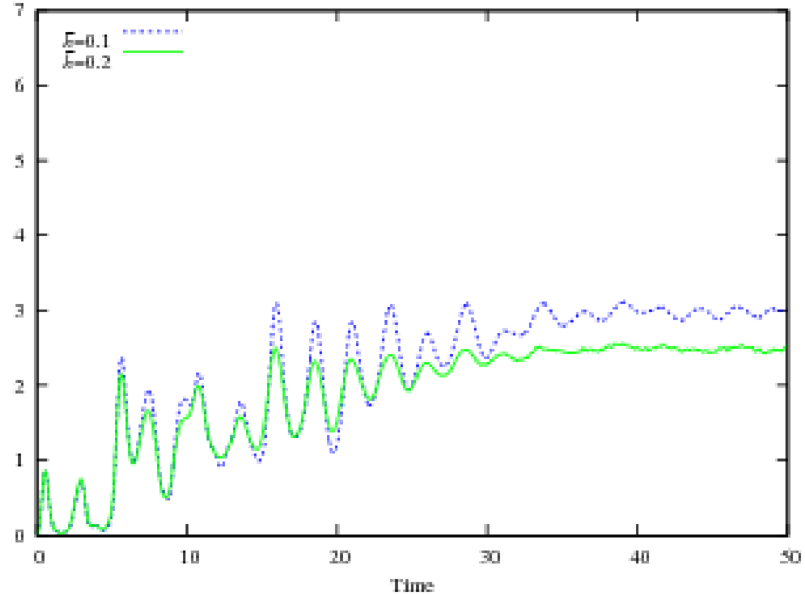


E=7

ħ= 0.1

ħ=0.2

Chaotic



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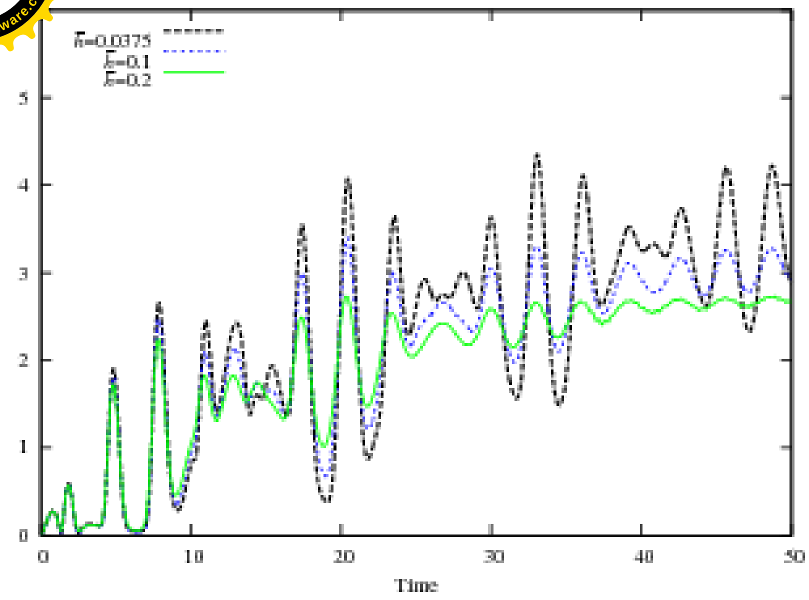
Integrable



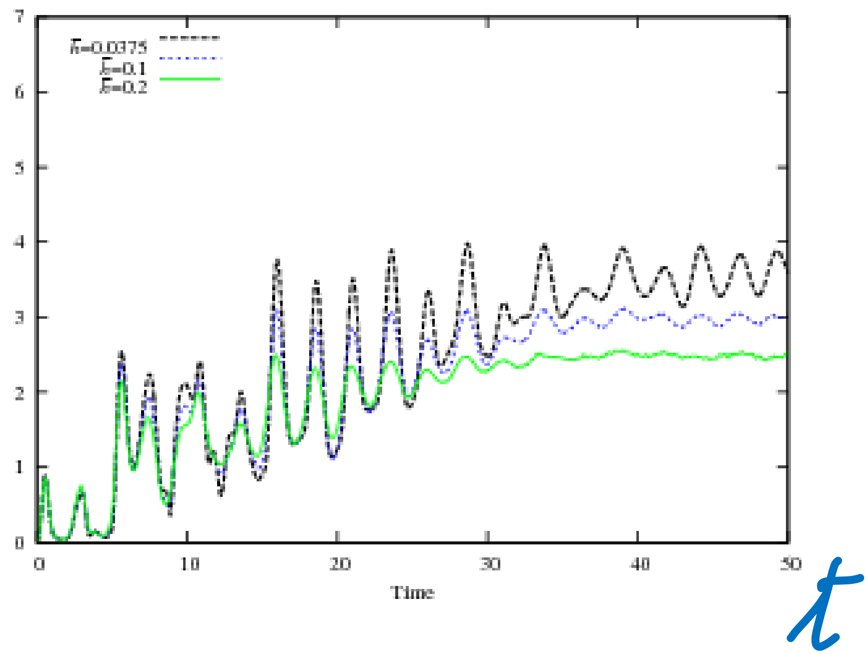
E=7

$\hbar$= 0.0375

$\hbar$= 0.1

$\hbar$=0.2

Chaotic



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The system
of two identical harmonic oscillators in a pure state

$$\frac{1}{\sqrt{2}} H_0 \otimes H_0 + \frac{1}{\sqrt{2}} H_1 \otimes H_1$$

H_0 **ground state**

H_1 **1° excited state**

$$S = -\sum_k \lambda_k^2 \ln \lambda_k^2 =$$
$$= \ln 2$$

Has entanglement entropy S = ln(2) independently on the value of $\hbar$ even though in the limit $\hbar \rightarrow 0$ its Wigner function tends to a Dirac δ function in the origin.

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$$\psi^{(0)}(x_i) \;=\; (\pi\hbar)^{-1/4} e^{-x_i^2/2\hbar}$$

$$\psi^{(1)}(x_i) \;=\; 2^{1/2}(\pi\hbar^3)^{-1/4} x_i e^{-x_i^2/2\hbar}$$

$$\boldsymbol{P} \;=\; |\Psi\rangle\langle\Psi|$$

$$\Psi(x_1, x_2) \;=\; \frac{1}{\sqrt{2}}\psi_1^{(0)} \otimes \psi_2^{(0)} \;+\; \frac{1}{\sqrt{2}}\psi_1^{(1)} \otimes \psi_2^{(1)}$$

$$\mathfrak{W}_{[P]} \;\to\; \delta(x_1)\delta(x_2)\delta(p_1)\delta(p_2)$$

Coarse-grained classical entropy

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We integrate classical eqs. for M=10^6 orbits starting at t=0, with the same initial conditions (gaussian density of initial points).

The four dim. ensemble thus obtained is projected onto reduced (q1,p1) plane

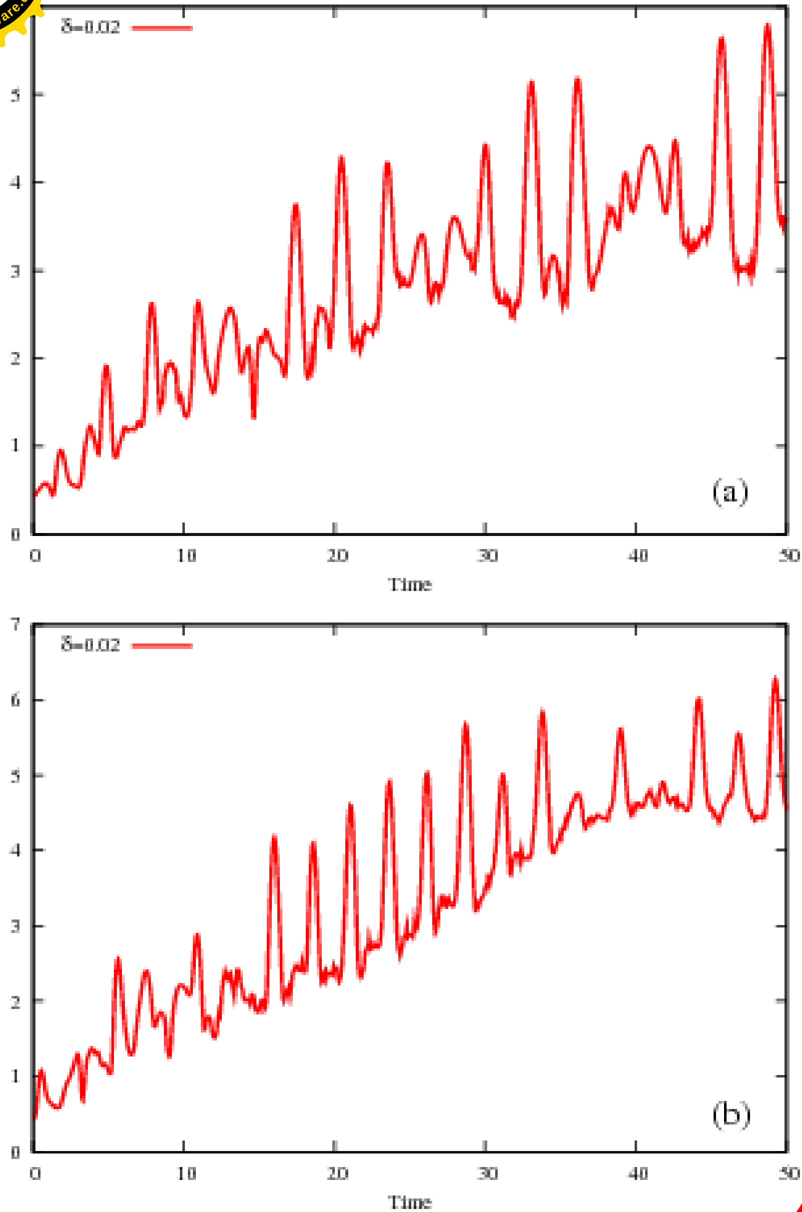
We divide the two- dim. phase space in square cells of size $\sqrt{\delta}$

w_i **is the number of orbits which fall in the cell i of the two-dimensional grid corresponding to particle 1, at time t.**

$$S_{\text{cl}}(\delta,t) = -\sum_i \frac{w_i(t)}{M} \ln \frac{w_i(t)}{M}$$

Coarse-grained classical entropy

S_{cl}



t

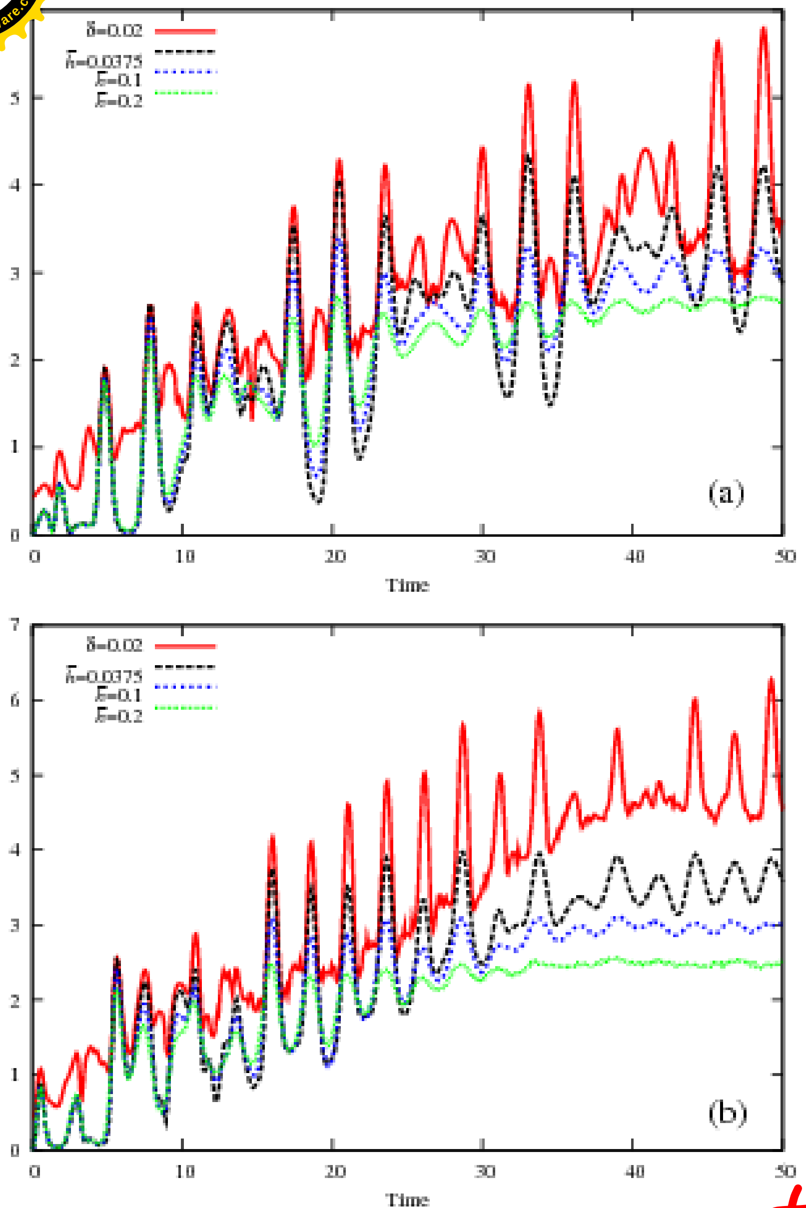
Integrable

E=7

$\delta = 0.02$

Chaotic

Integrable



E=7

classical

ħ= 0.0375

ħ= 0.1

ħ=0.2

Chaotic

t

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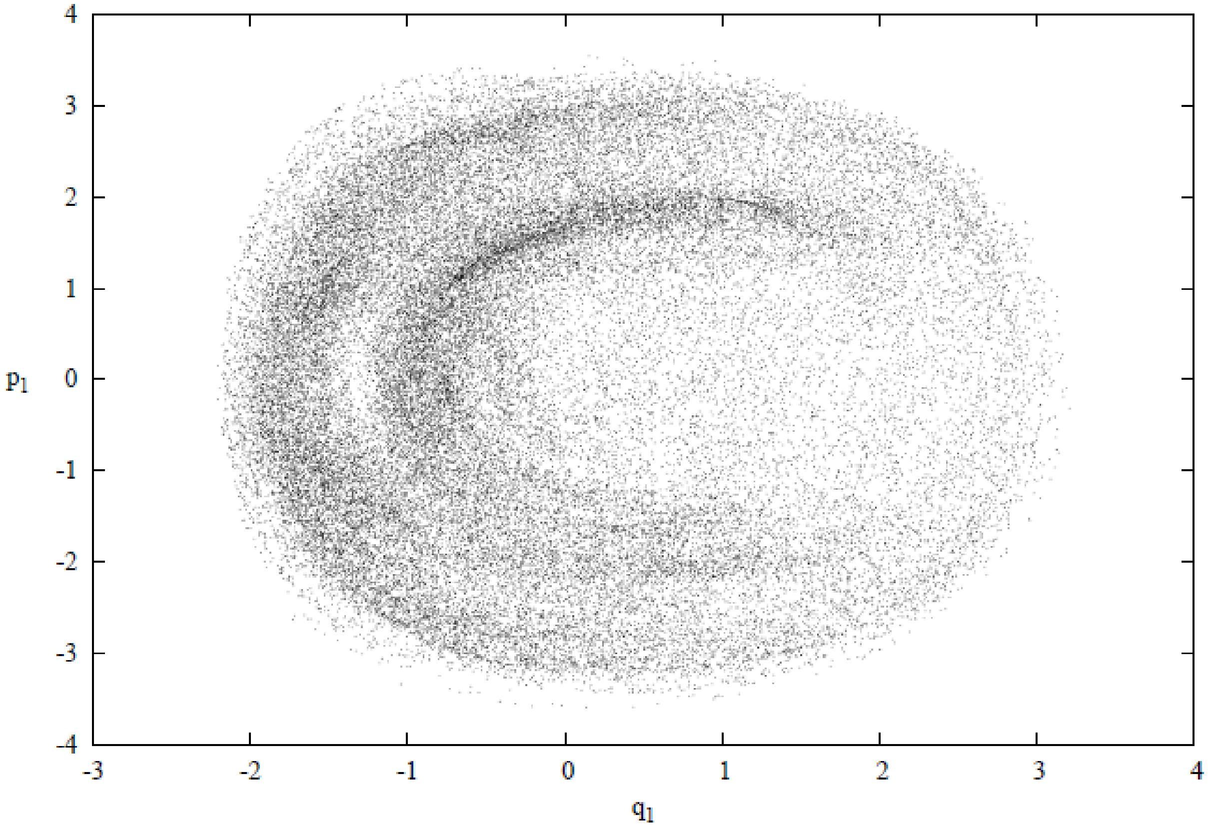
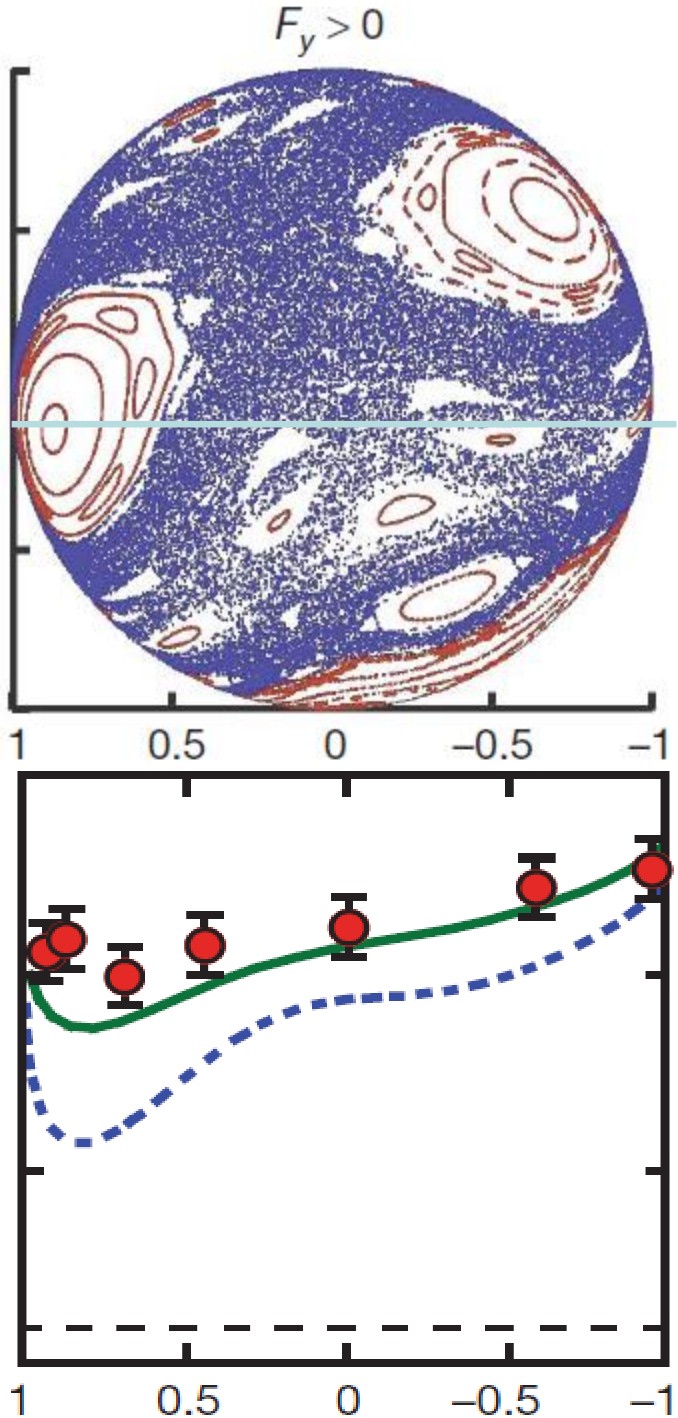


FIG. 5: Projected p_1, q_1 ensemble for the chaotic case with $E = 7$, $m_2/m_1 = 0.54$, at time $t = 200$.

the projection of each invariant torus on the ***(q1, p1) plane*** **has caustics at its boundaries, because there the tangent plane of the torus is "vertical**



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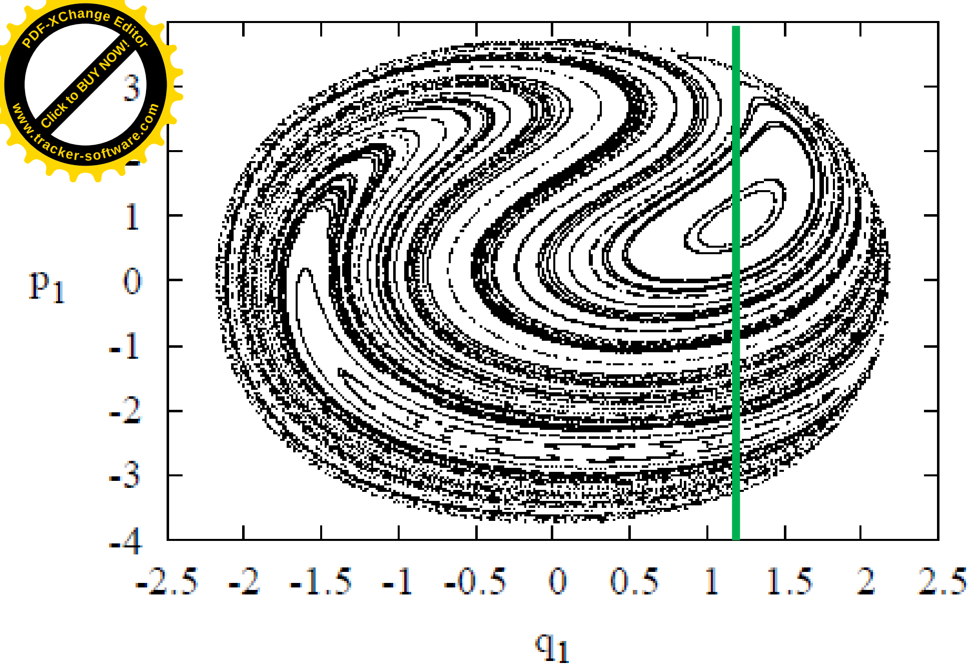

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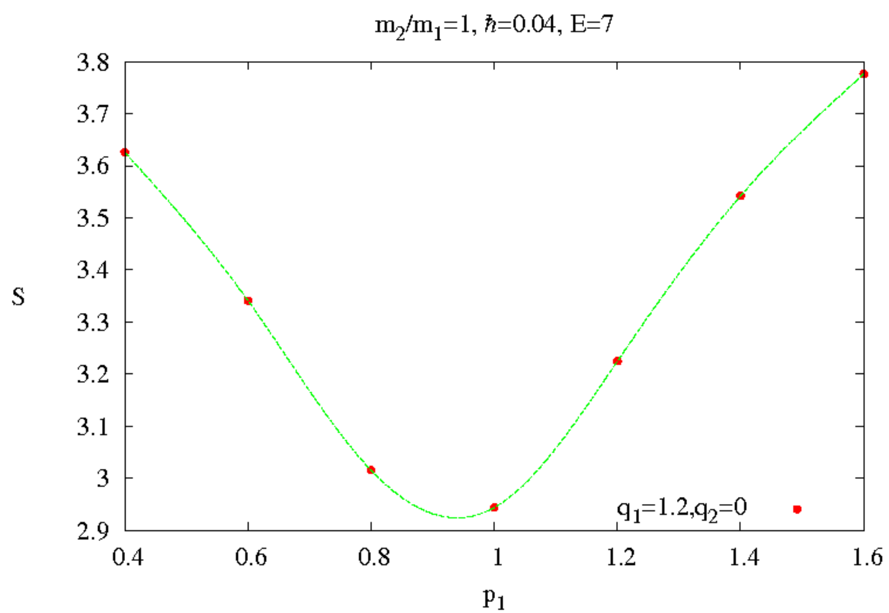

Entanglement is Necessary for Emergent Classicality in All Physical Theories

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(Received 11 October 2016; revised manuscript received 10 June 2017; published 24 August 2017)

entangled states, or is quantum theory special? One important feature of quantum theory is that it has a classical limit, recovering classical theory through the process of decoherence. We show that any theory with a classical limit must contain entangled states, thus establishing entanglement as an inevitable feature of any theory superseding classical theory.

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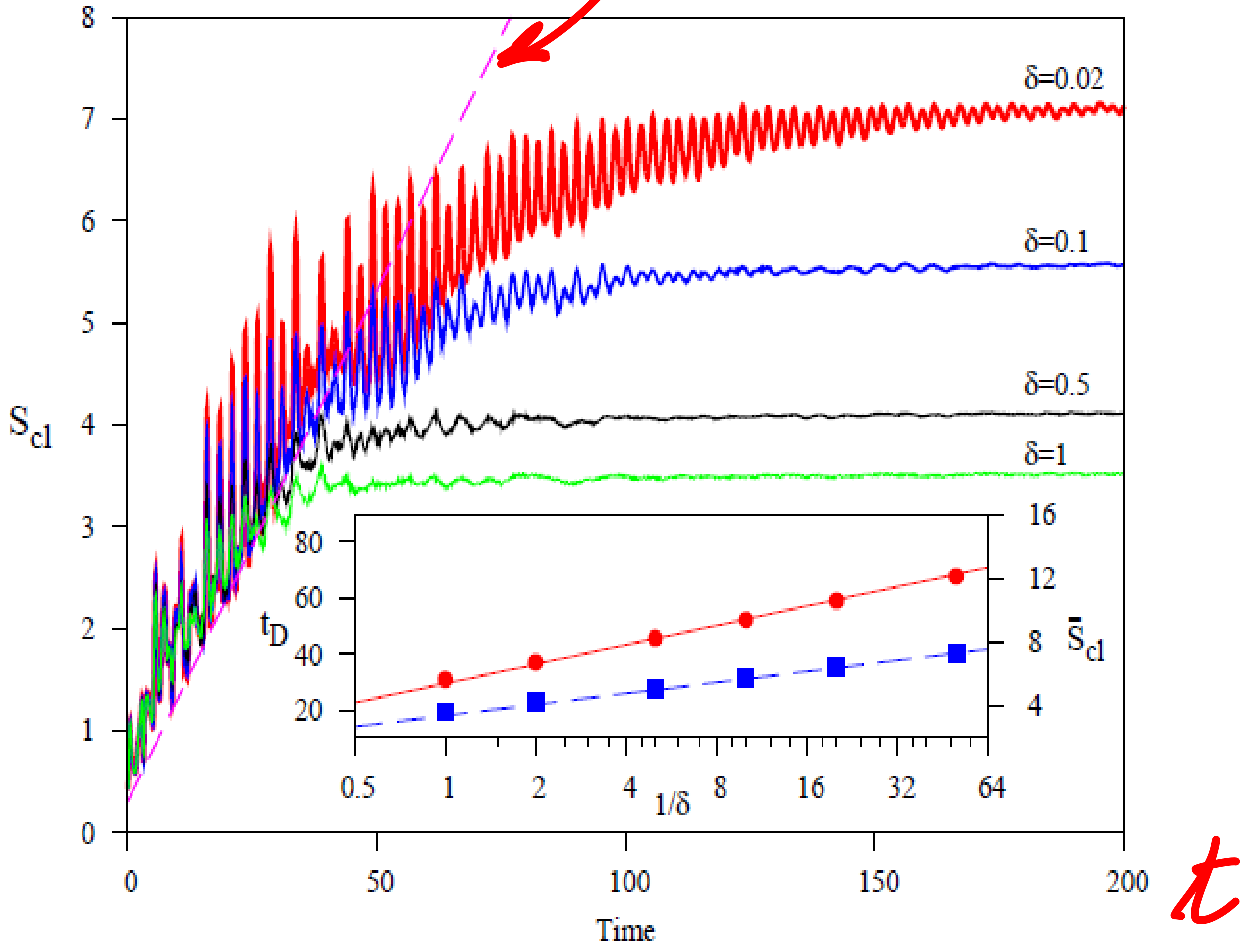
Saturation values: $\sim -\ln(\delta)$

Deviation times: $\sim -10\ln(\delta)$

$S_{cl} \sim t/10$



Chaotic case

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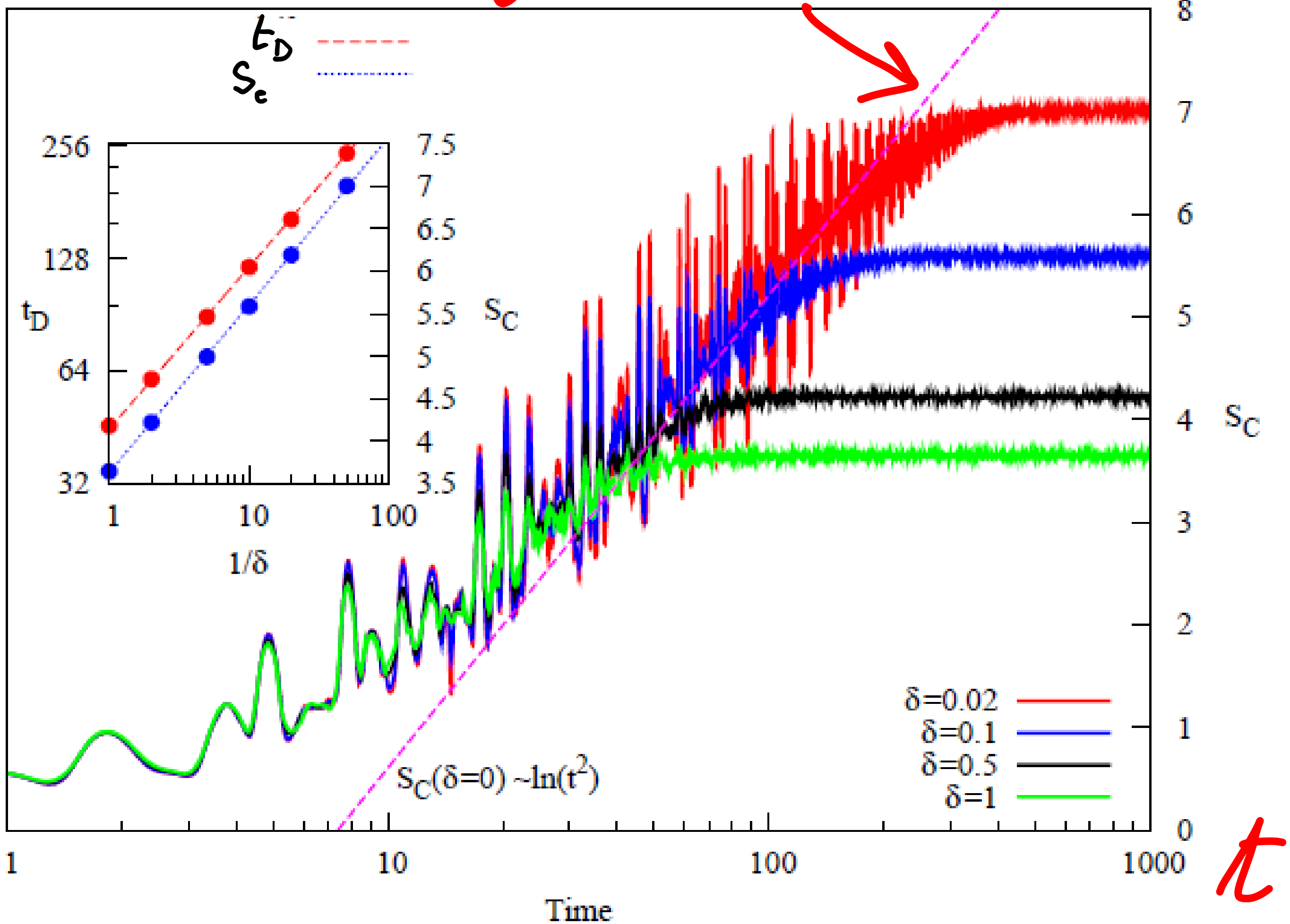
Saturation values: $\sim -\ln(\delta)$

Deviation times: $\sim 1/\sqrt{\delta}$

$S_{cl} \sim 2 \ln t$

Integrable case



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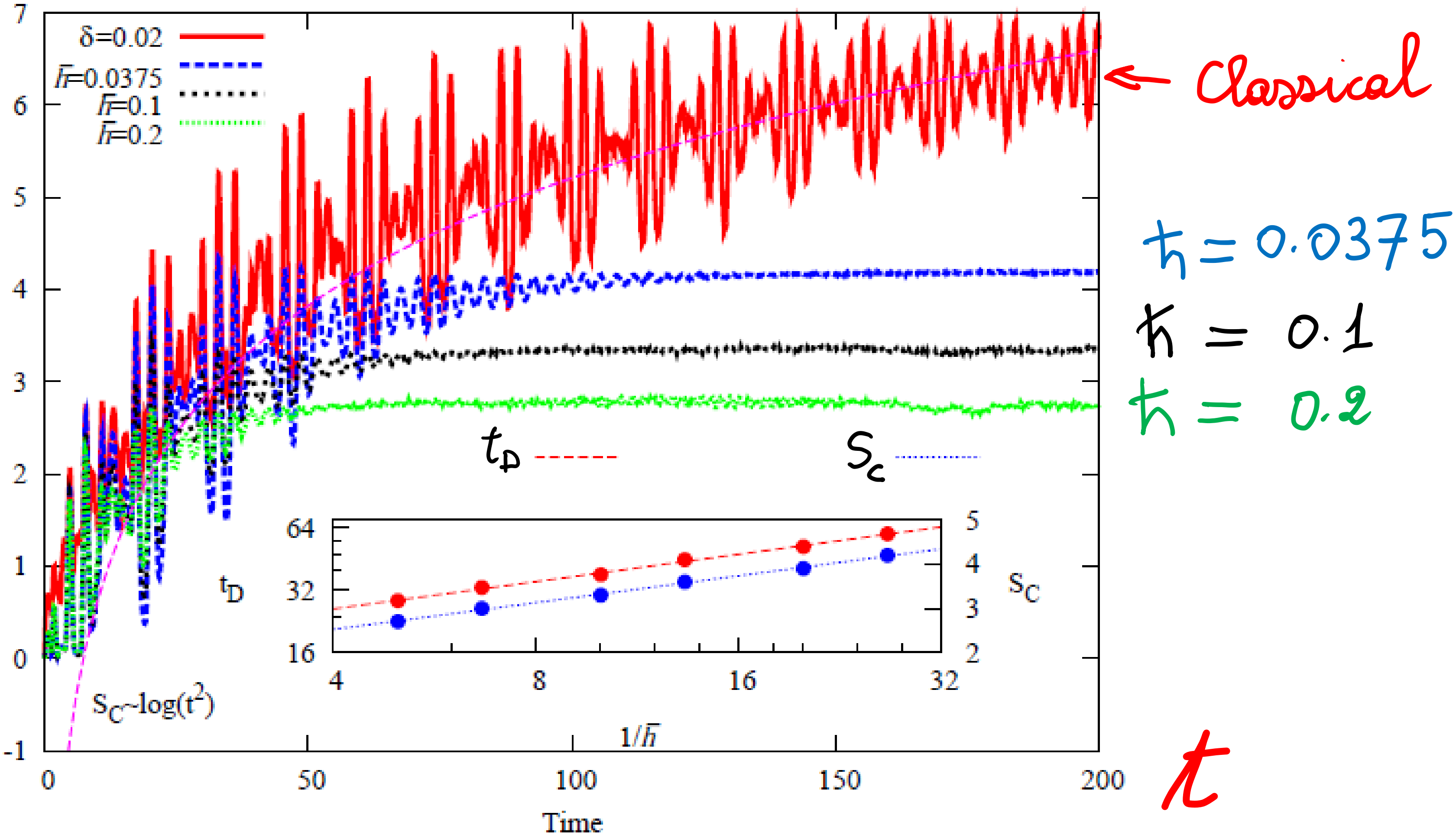
Saturation values: $\sim -\ln(\hbar)$

Deviation times: $\sim 1/\sqrt{\hbar}$

Integrable case



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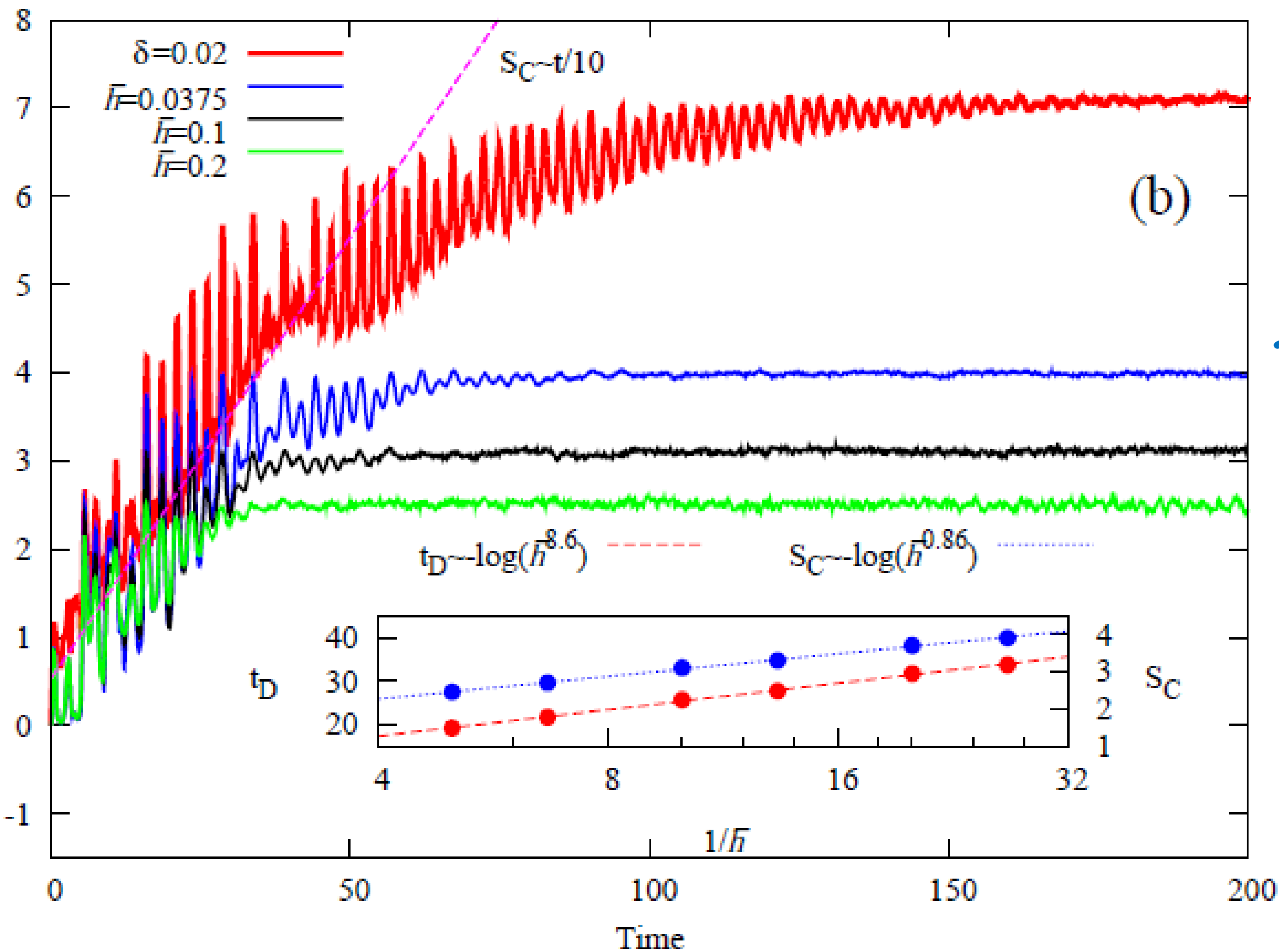
Saturation values: $\sim -\ln(\hbar)$

Deviation times: $\sim -10\ln(\hbar)$

Chaotic case



Classical

$\hbar = 0.0375$

$\hbar = 0.1$

$\hbar = 0.2$

t

Coupled oscillators

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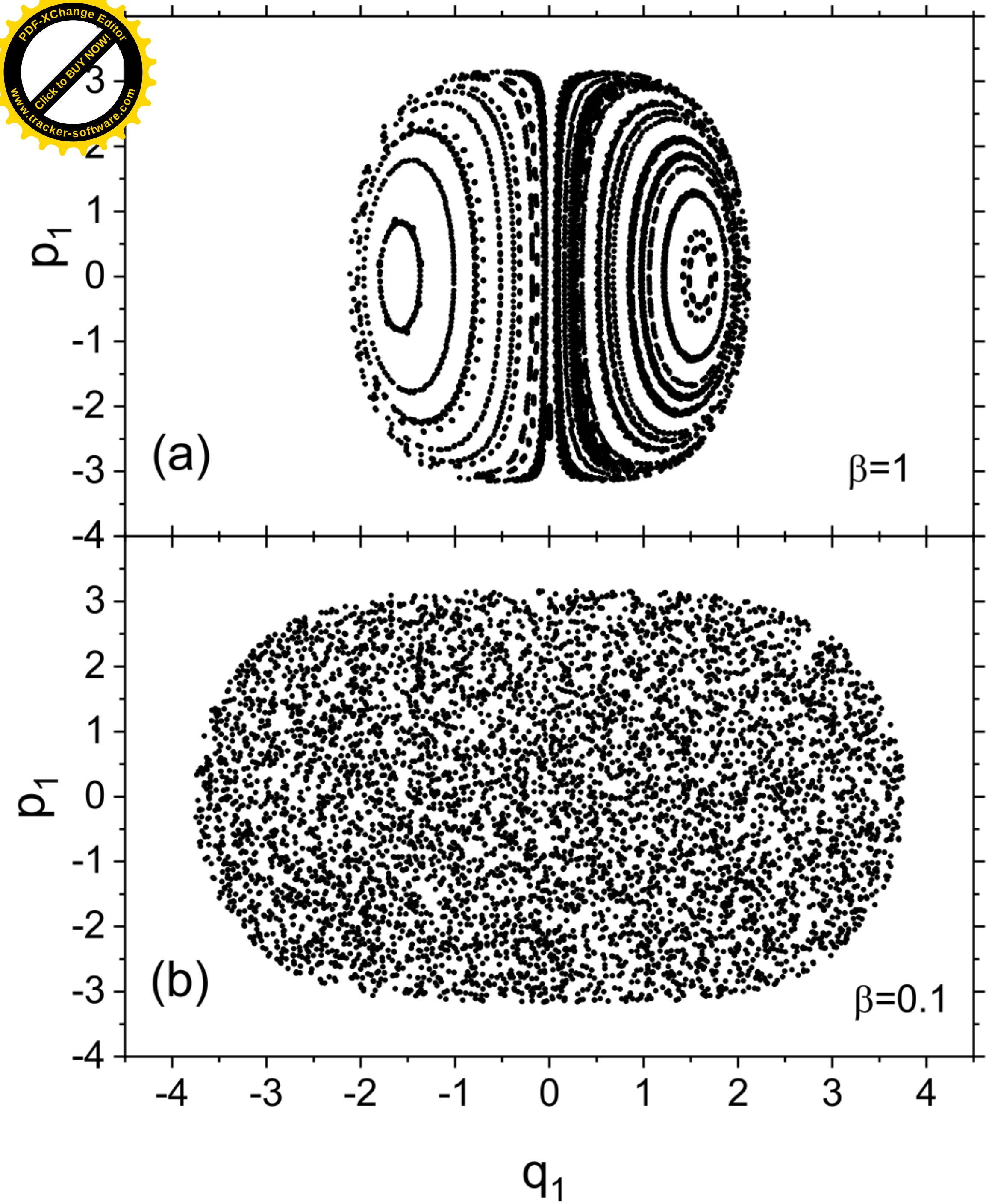
$$H_C = \frac{1}{2}(p_1^2 + p_2^2) + \frac{\beta}{4}(q_1^4 + q_2^4) + \frac{1}{2}q_1^2 q_2^2.$$

Integrable **for** $\beta = 1 \text{ and } \beta = \frac{1}{3}$

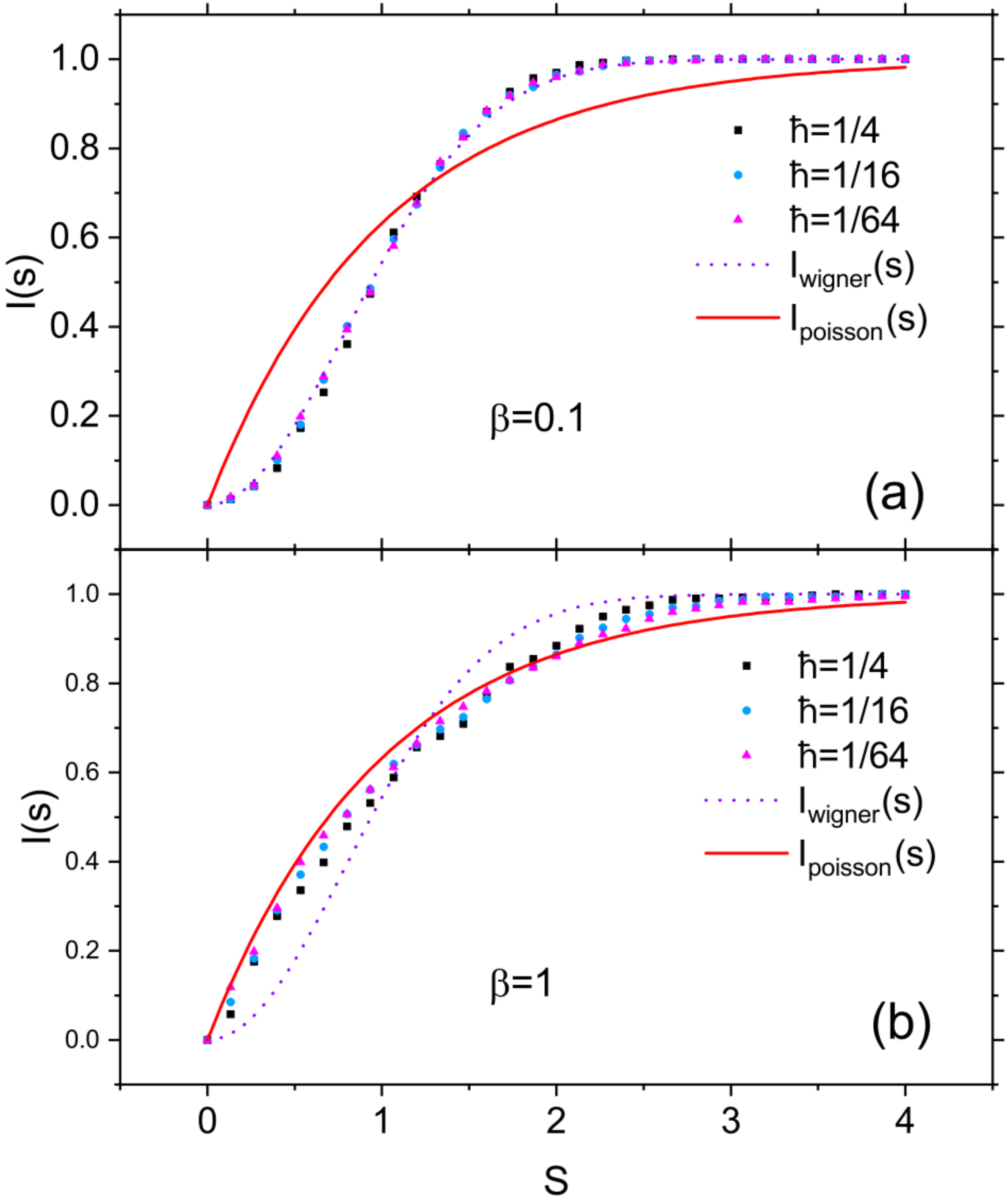
Chaotic **when** β **close to zero**



Surface of section

$q_2 = 0$

$E = 5$



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Initial state

Quantum

Direct product of coherent states

$$\psi(x_1, x_2, 0) = \langle x_1 x_2 | \psi_0 \rangle = \psi_1(x_1, 0) \otimes \psi_2(x_2, 0)$$

$$\psi_j(x_j, 0) = \exp\left(-\frac{(x_j - x_j^0)^2}{2\hbar} - \frac{i p_j^0 x_j}{\hbar}\right)$$

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Classical

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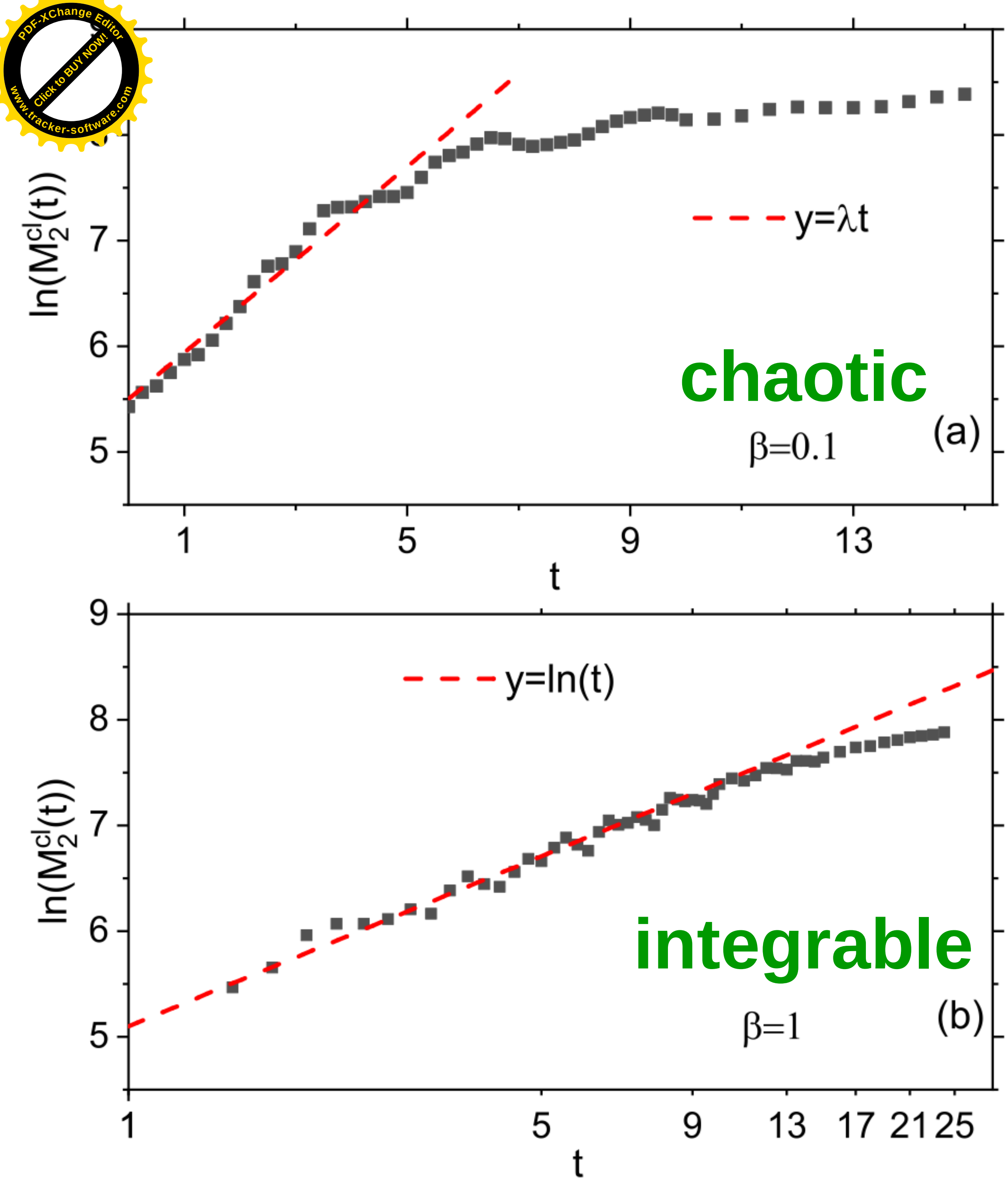
Gaussian distibution

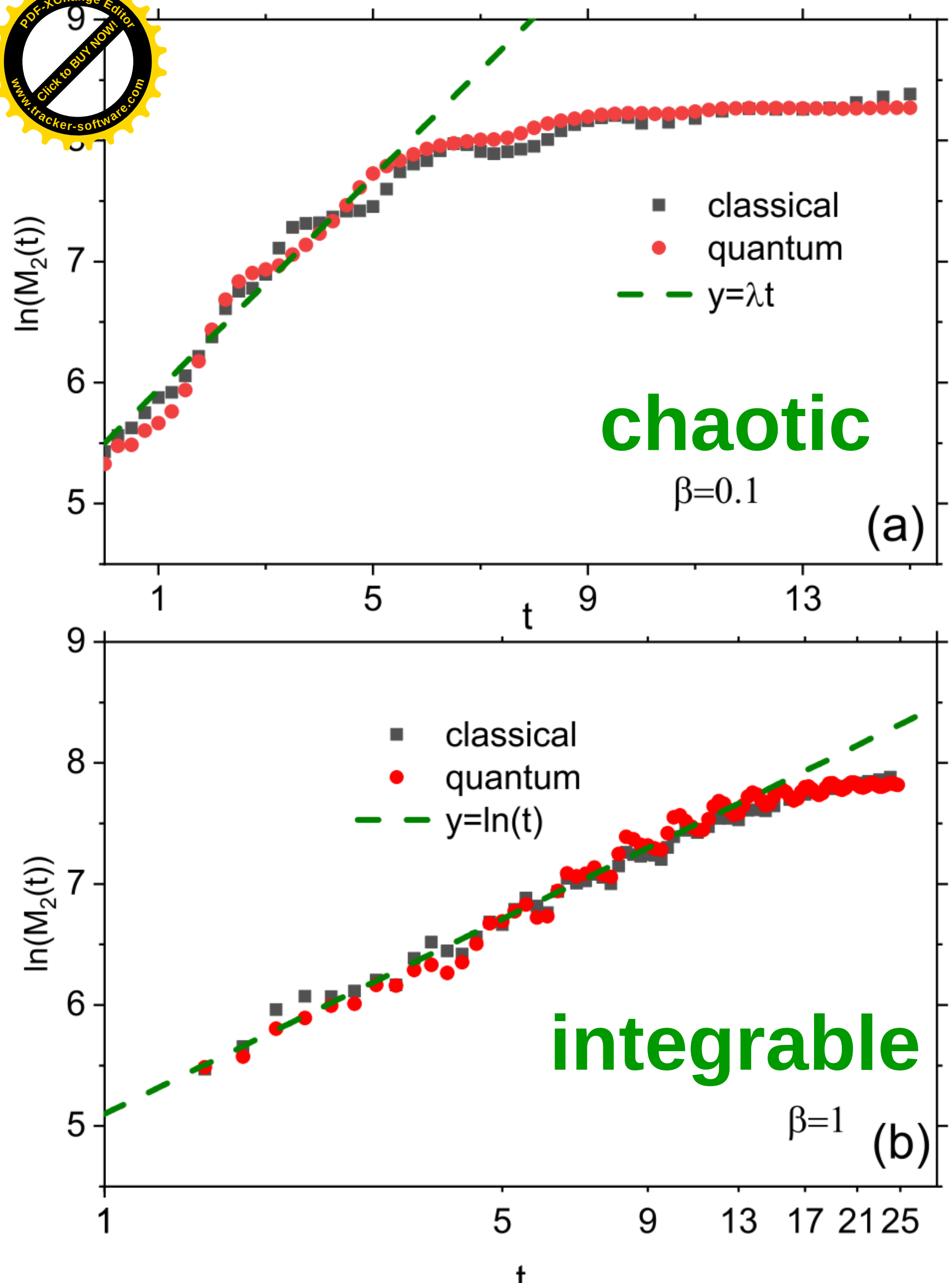
$$\rho(x_1, p_1, x_2, p_2, 0) = \rho_1(x_1, p_1, 0)\rho_2(x_2, p_2, 0)$$

$$\rho_j(x_j, p_j, 0) = \exp(-\frac{(x_j - x_j^0)^2}{\hbar} - \frac{(p_j - p_j^0)^2}{\hbar})$$

Results averaged over several initial distributions



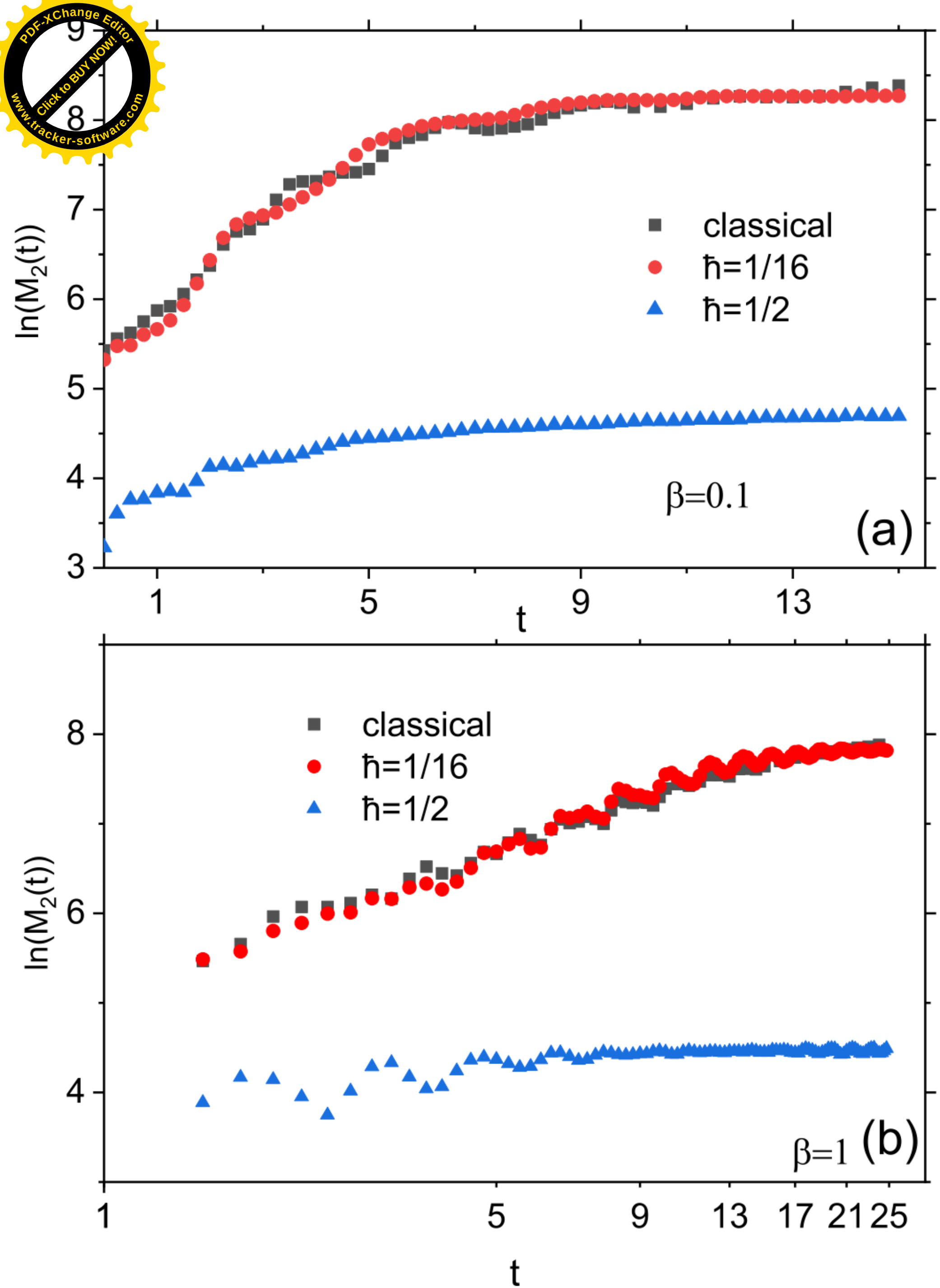
Results are averages over 50 initial ensembels with center randomly distributed on the energy surface



Classical

quantum

$\hbar = 1/16$



■ Classical
● ħ = 1/16
▲ ħ = 1/2

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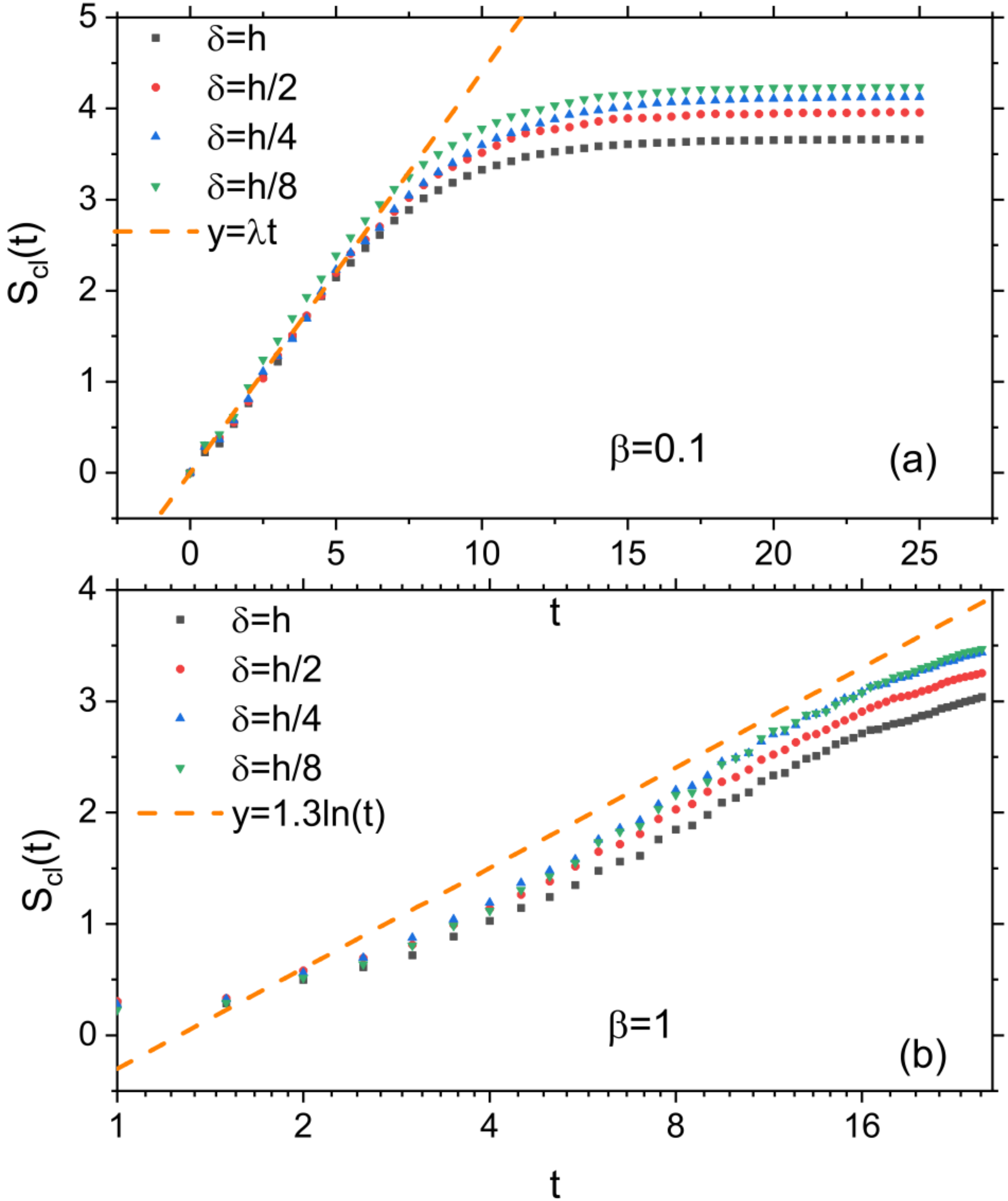

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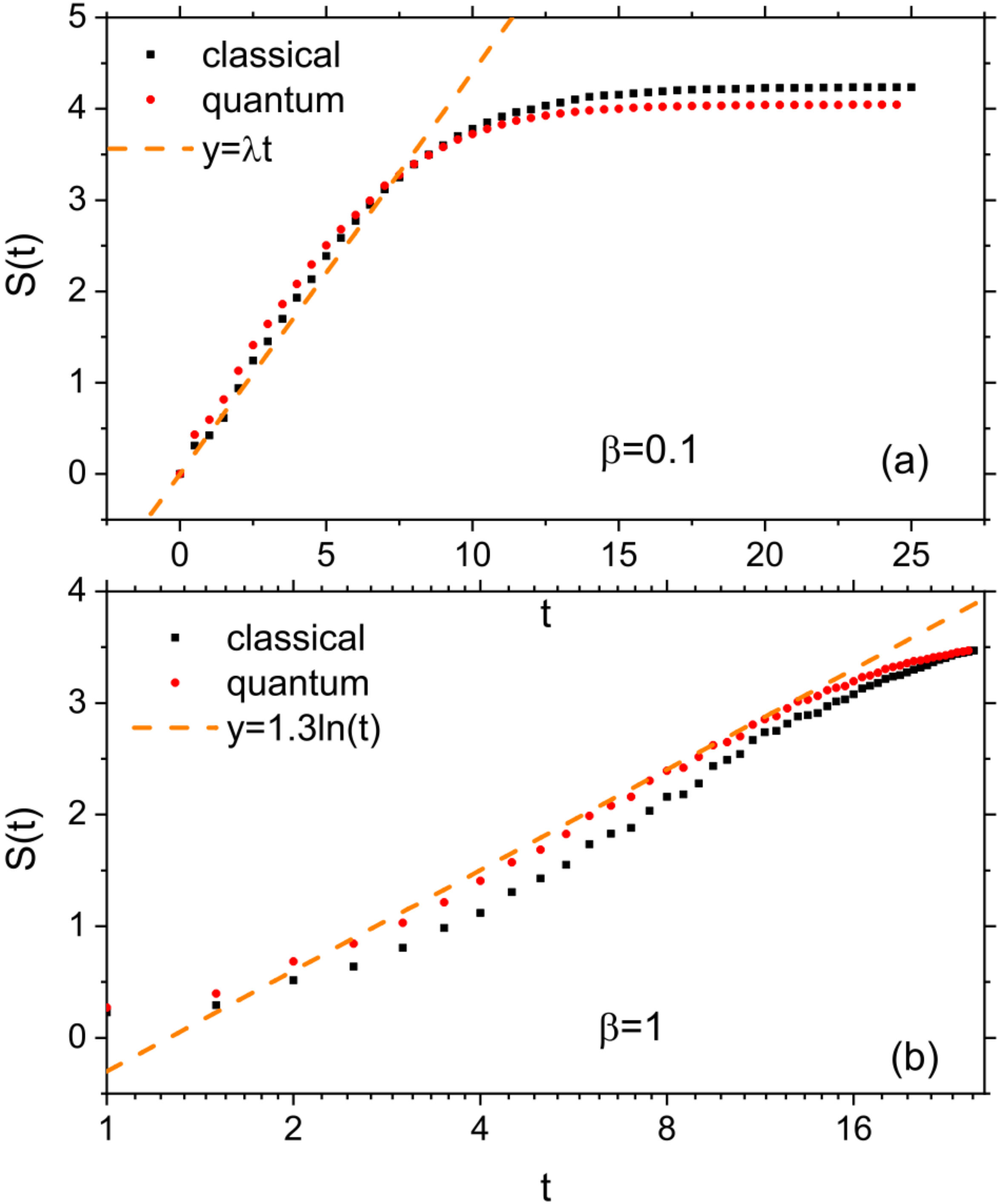

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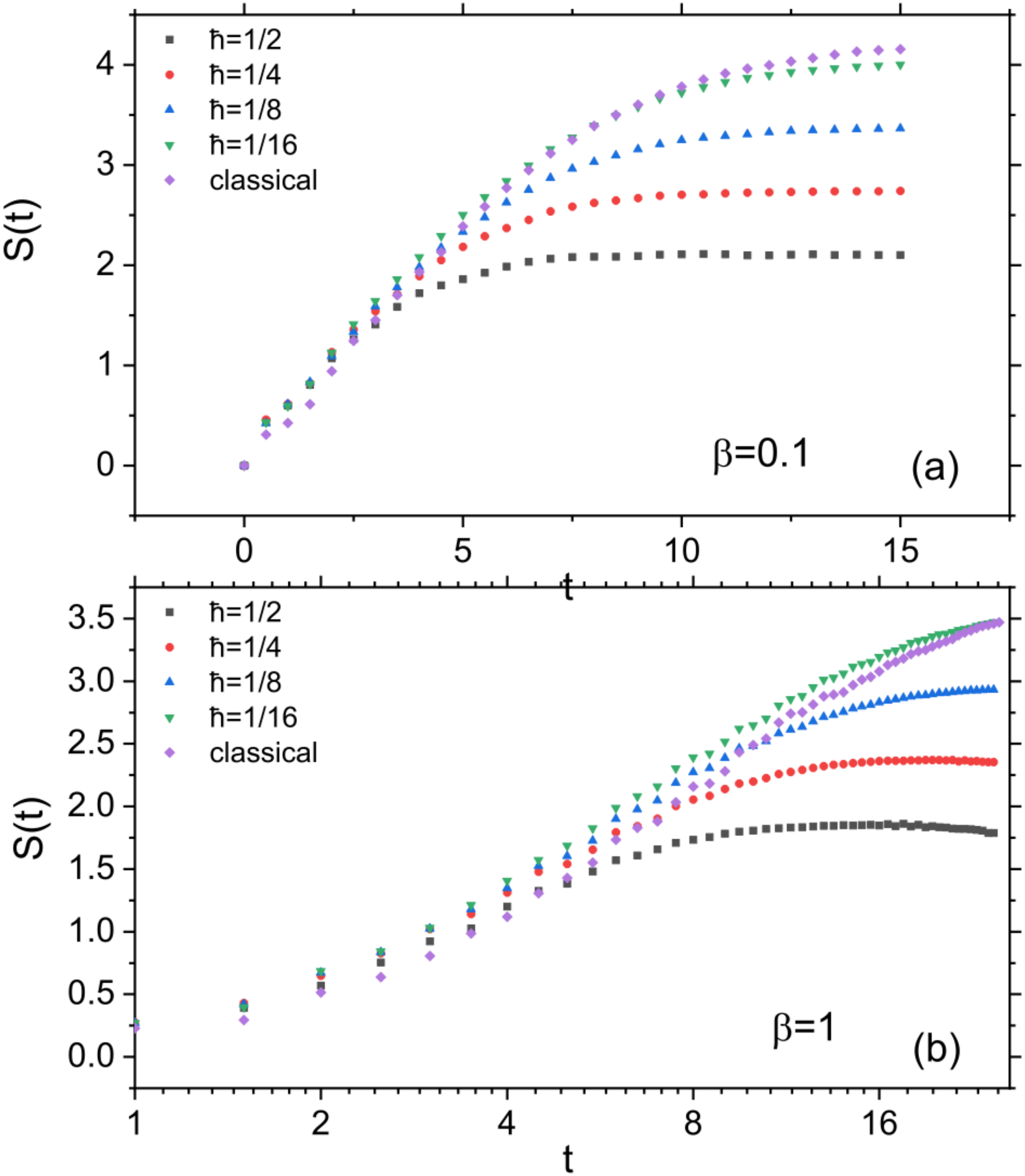
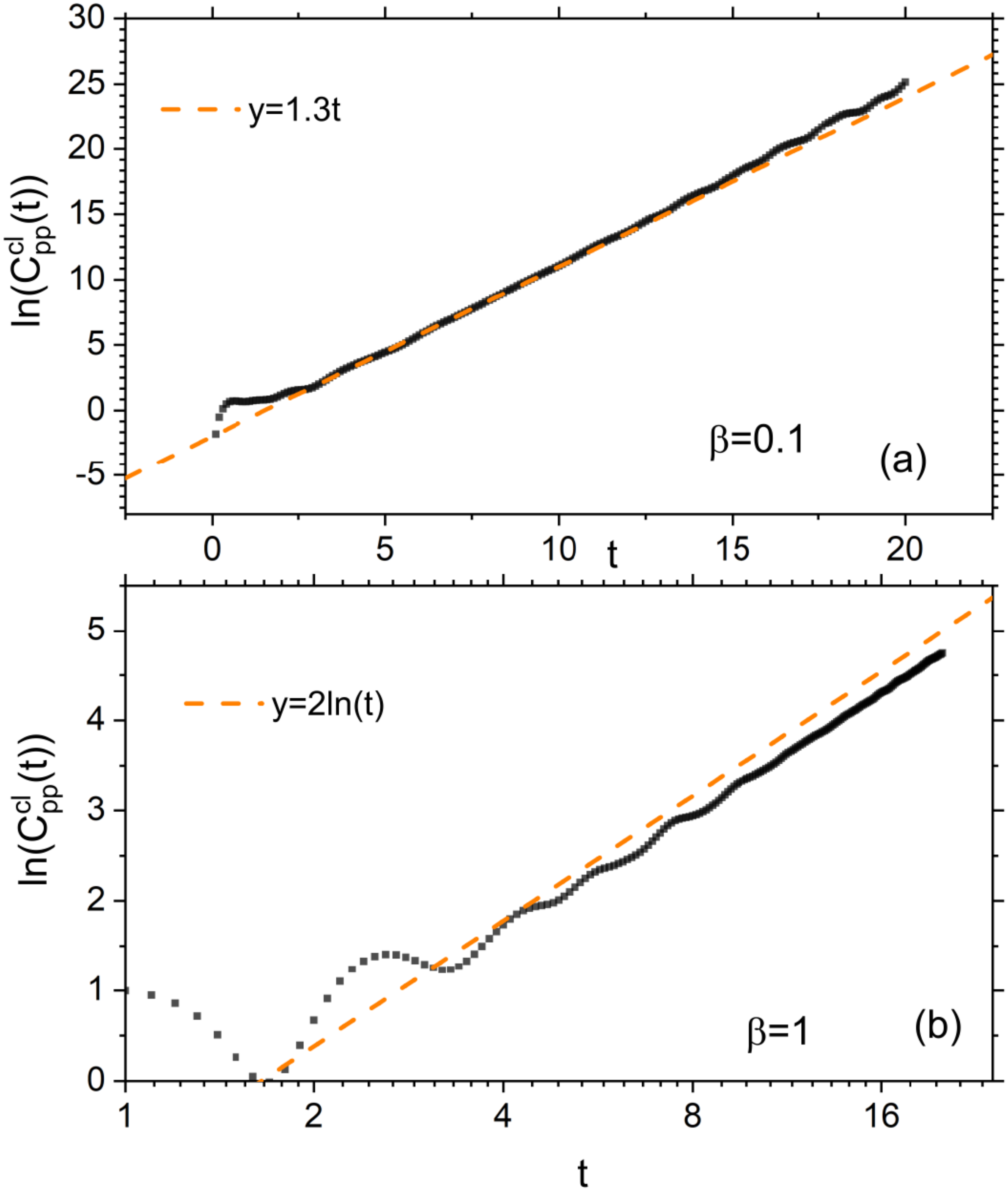
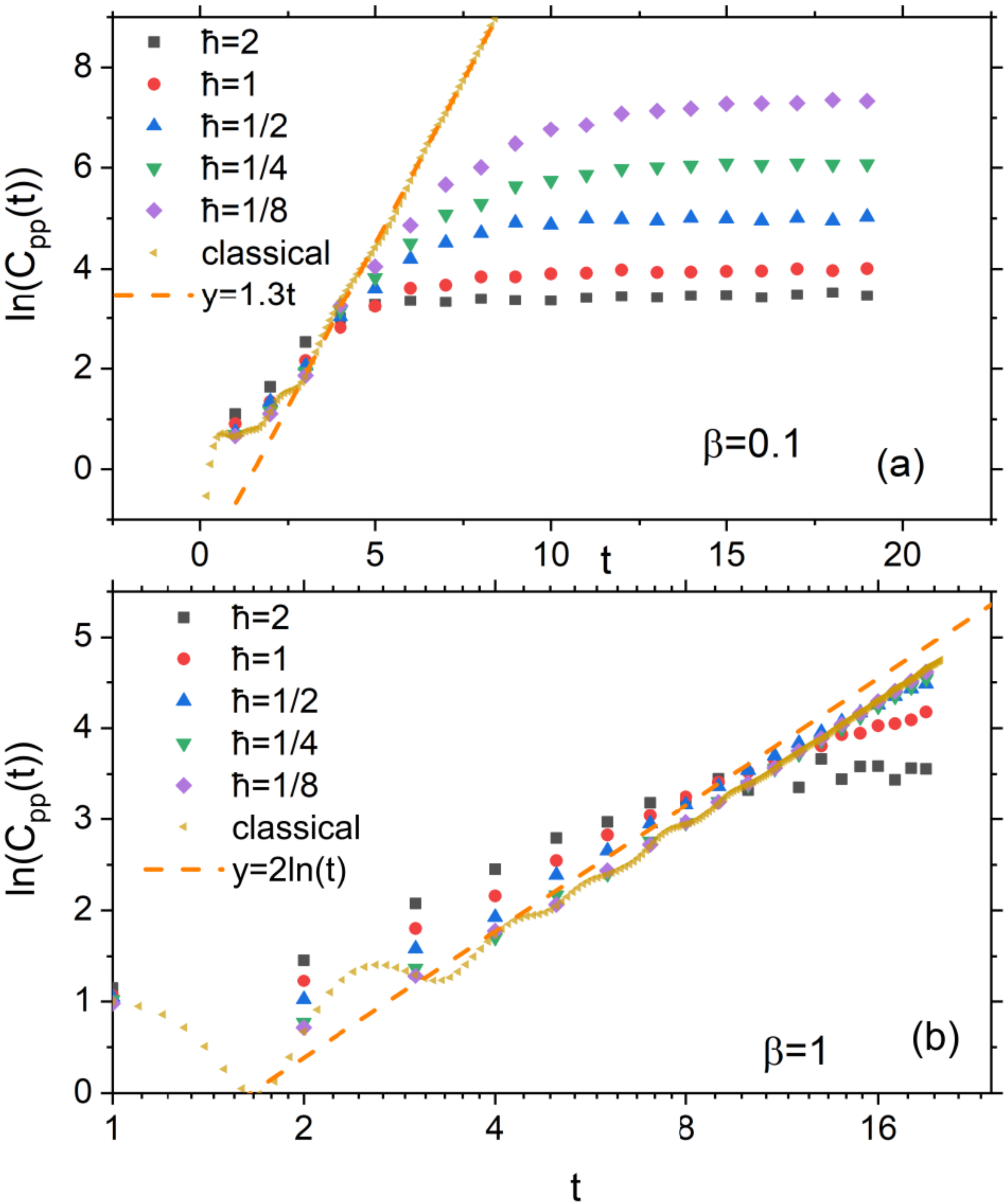


$$C_{pp}(t) = -\frac{1}{\hbar^2}\langle\psi_0|[p_1(t),p_1(0)]^2|\psi_0\rangle$$

$$C^{cl}_{pp}(t) \simeq \{p_1(t),p_1(0)\}^2_{poisson} \quad =$$

$$= \left\langle \left(\frac{\partial p_1(t)}{\partial q_1(0)}\right)^2 \right\rangle_0 \simeq \left\langle \left(\frac{\delta p_1(t)}{\delta q_1(0)}\right)^2 \right\rangle_0$$


y=1.3t
ln(C^cl_pp(t))
β=0.1
(a)
t
y=2ln(t)
ln(C^cl_pp(t))
β=1
(b)
t



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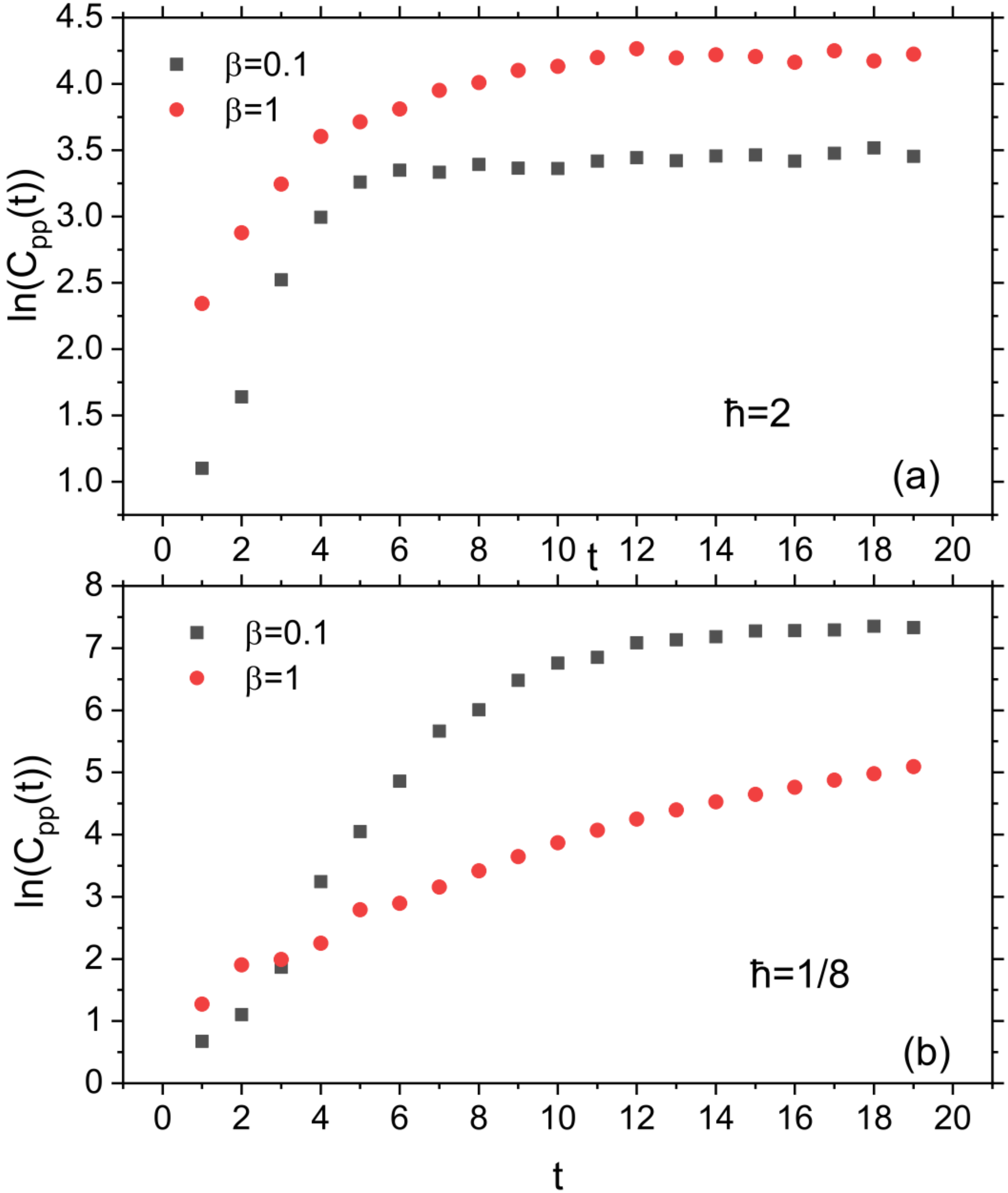

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